

A strict undirected model for the k -nearest neighbour graph

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Abstract

Let $G = G_{n,k}$ denote the graph formed by placing points in a square of area n according to a Poisson process of density 1 and joining each pair of points which are both k nearest neighbours of each other. Then $G_{n,k}$ can be used as a model for wireless networks, and has some advantages in terms of applications over the two previous k -nearest neighbour models studied by Balister, Bollobás, Sarkar and Walters, who proved good bounds on the connectivity models thresholds for both. However their proofs do not extend straightforwardly to this new model, since it is now possible for edges in different components of G to cross. We get around these problems by proving that near the connectivity threshold, edges will not cross with high probability, and then prove that G will be connected with high probability if $k > 0.9684 \log n$, which improves a bound for one of the models studied by Balister, Bollobás, Sarkar and Walters too.

1 Introduction

Let $G_{n,k}$ be the graph formed by placing points in S_n , a $\sqrt{n} \times \sqrt{n}$ square, according to a Poisson process of density 1 and connecting two points if they are both k -nearest neighbours of each other (i.e. one of the k -nearest points in S_n). We will refer to this as the strict undirected model. A natural question, especially when considering this as a model for a wireless network, is: Asymptotically, how large does k have to be in order to ensure that $G_{n,k}$ is connected?

We cannot ensure with certainty that the resulting graph will be connected; there will always be a chance that a local configuration will occur that produces multiple components, but we can ask: what value of k ensures that the probability of the graph being connected tends to one? Indeed we say that $G_{n,k}$ has a property Π *with high probability* if $\mathbb{P}(G_{n,k} \text{ has } \Pi) \rightarrow 1$ as $n \rightarrow \infty$. So we seek to answer the question: What $k = k(n)$ ensures that $G_{n,k}$ is connected with high probability?

Different variations of this problem have been studied previously, using different connection rules. Gilbert [4] first introduced a model in which every point was joined to every other point within some fixed distance, R (the Gilbert model). Equivalently, this can be viewed as joining each point, x , to every point within the circle of area πR^2 centred on x . Penrose proved in [5], that

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if $\pi R^2 \geq (1 + o(1)) \log n$ (so that on average each point is joined to at least $\log n$ other points), then the resulting graph is connected with high probability, whereas if $\pi R^2 \leq (1 + o(1)) \log n$, then the resulting graph is disconnected with high probability.

Xue and Kumar [7] studied the model in which two points are connected if either is the k -nearest neighbour of the other (we will denote this graph $G'_{n,k}$), and proved that the threshold for this model is $\Theta(\log n)$. Balister, Bollobás, Sarkar and Walters [1] considerably improved their bounds (they showed that if $k < 0.3043 \log n$ then $G'_{n,k}$ is disconnected whp, while if $k > 0.5139 \log n$ then $G'_{n,k}$ is connected whp). In the same paper, Balister, Bollobás, Sarkar and Walters also examined a directed version of the problem where a vertex sends out an out edge to all of its k nearest neighbours, and again showed that the connectivity threshold is $\Theta(\log n)$ obtaining upper and lower bounds of $0.7209 \log n$ and $0.9967 \log n$ respectively.

It has been pointed out that for practical uses (e.g. for wireless networks), it would be better to use a different connection rule, namely to connect two points only if they are both k nearest neighbours of each other. This model has two advantages in terms of wireless networks: It ensures that no vertex will have too high a degree, and thus be swamped, as could happen with either of the previous models. It also ensures we can always receive an acknowledgement of any information sent at each step, which may not be the case in the directed model.

The edges in our new model are exactly the edges in the directed model which are bidirectional, and so any lower bound proved for the directed model will also be a lower bound for the strict undirected model. Thus, from Balister, Bollobás, Sarkar and Walters [1] we know that if $k < 0.7209 \log n$ then $G_{n,k}$ is disconnected with high probability. It can be shown using a tessellation argument and properties of the Poisson process, that the connectivity threshold in this model is again $\Theta(\log n)$ (e.g. see the introduction of [1]), and so our task is to produce a good constant, c , for the upper bound such that if $k > c \log n$ then $G_{n,k}$ is connected with high probability. In particular we will show that some $c < 1$ will do, to show that a conjecture of Xue and Kumar made for the original undirected model [7] (and which is true for the Gilbert model) does not hold for this model. The method used in [1] for both of the previous models was to show first that for any $c' > 0$, if $k > c' \log n$ then there could be only one ‘large’ component of $G_{n,k}$ with high probability. This allowed them to concentrate on ‘small’ components, and so gain their bounds.

We wish to do the same, however our model has some extra complications. One key property used in the proofs that there is only one large component was that edges in different components of G cannot cross, but that is not the case in the strict undirected model. Indeed, Figure 1 shows the outline of a construction in which the edges of two different components do cross.

Luckily, the set-up required for edges of different components to cross is fairly restrictive, and we are able to show:

Theorem 1. *If $k = c \log n$, then, for $c > 0.7102$ (and in particular below the connectivity threshold), no two edges in different components inside G will cross with high probability.*

Remark. *Officially this should read “If $k = \lceil \log n \rceil$, then...,” however, since we are considering the limit as n tends to infinity, this makes no difference, and*

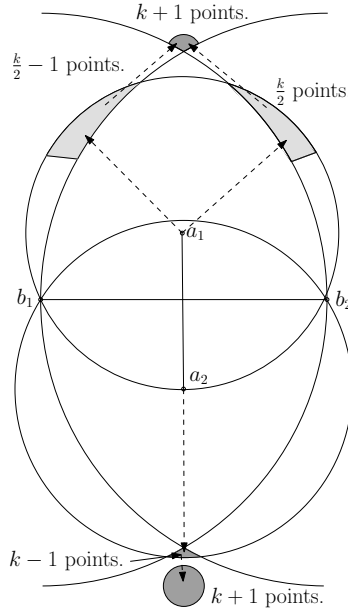


Figure 1: If each of the shaded regions has the number of points shown, and there are no other points nearby, then a_1a_2 and b_1b_2 would be edges of $G_{n,k}$, but a_1 and a_2 would be in a different component from b_1 and b_2 (Here dashed arrows indicate directed out edges between regions).

so for ease of notation we leave the ceiling notation out here, and for the rest of the paper.

There are further complications in proving good upper bounds on the connectivity threshold: In both of the previous models it was always the case that if there was no edge from a point x to a point y , then there must be at least k points closer to x than y is, whereas in our model we may only conclude that one or the other has k nearer neighbours. For this reason we have to handle the case of small components differently too. We are able to show:

Theorem 2. *If $k = c \log n$ and $c > 0.9684$, then G is connected with high probability.*

We first introduce some basic definitions and notation that will be used throughout the paper.

2 Notation and Preliminaries

Definition 1. *Given a point $a \in G_{n,k} = G$, we write $\Gamma^+(a)$ for the set of the k -nearest neighbours of a and define this to be the out neighbourhood of a . We define the k -nearest neighbour disk of a , denoted $D^k(a)$, to be the smallest disk centred on a that contains $\Gamma^+(a)$.*

We will often say that a point x has an out edge to a point y (or that \vec{xy} is an out edge) to mean that $y \in \Gamma^+(x)$. Note that xy is an edge in G if and

only if both \overrightarrow{xy} and \overrightarrow{yx} are out edges. Correspondingly we say that x has an in edge from y if \overrightarrow{yx} is an out edge.

We will use the following notational conventions:

- We write $D_a(r)$ for the disk of radius r centred on a .
- We will use capital letters to represent sets (e.g. a region of the plane, or a component), and lower case letters for points in the plane (however if a and b are points, we will write ab for the edge (straight line segment) from a to b).
- For two sets A and B , we write $d(A, B)$ for the minimum distance from any point in A to any point in B . For a point x and a region B we write $d(x, B) = d(\{x\}, B)$.
- For a set A , we write ∂A for the boundary of the closure of A .
- Given a region A , we write $\#A$ for the number of points of G in A , and $|A|$ for the area of A . We write $\|ab\|$ for the length of the edge ab .
- We will refer to the vertices of G as *points* (i.e. points of our Poisson process), and a single element of S_n as a *location*.
- We will often introduce Cartesian co-ordinates onto S_n (with scaling), and when this is the case, we will write $p^{(x)}$ and $p^{(y)}$ for the x and y co-ordinates of any point/location p .

At times we will refer to specific points and regions of G and S_n , especially in the proof that edges of different components cannot cross (Section 3.2), and so to help keep things easy to follow, a list of definitions and notations is included in Appendix A.

3 Edges of different components cannot cross, and there can only be one large component

The eventual aim of this section will be to show that if $c = 0.7102$ and $k > c \log n$, then with high probability there will only be one large component. We will achieve this by bounding the minimal distance between two edges in different components of G . As a first step we establish a lower bound on the distance of a point of G and an edge in a different component.

3.1 Preliminaries - An edge of one component cannot be too close to a vertex in another component

To prove a bound on the distance between a point of G and an edge in a different component, we first state the following result of Balister, Bollobás, Sarkar and Walters [1] that bounds how close points in different components of G can be. This lemma was proved for the original undirected model, but the proof uses properties of the Poisson process only. Namely, they showed that, given a point x , for any point y that is close enough to x we will have $\mathbb{P}(\overrightarrow{xy} \text{ not an out edge}) = O(n^{1-\varepsilon})$, and thus that with high probability all points

close enough together have out edges to each other. Since this implies \overrightarrow{xy} and \overrightarrow{yx} are both out edges for x and y close enough together, it also shows that xy would be an edge in our model.

Lemma 3. *Fix $c > 0$, and set;*

$$c_- = ce^{-1-1/c} \text{ and } c_+ = 4e(1+c)$$

If r and R are such that $\pi r^2 = c_- \log n$ and $\pi R^2 = c_+ \log n$, then whp every vertex in $G_{n,k}$ is joined to every vertex within distance r , and every vertex has at least $k+1$ other vertices within a distance R , and so in particular is not joined to any vertex more than a distance R away.

The next lemma will be used repeatedly, and is a result about how points can be connected in our graph. It states that the longest edge (in G) out of any point, x , is at most twice the shortest non-edge involving x , or, equivalently, that the region containing the neighbourhood of x (in G) is at most a factor of two off being circular. This is certainly not the case in either of the two previous models.

Lemma 4. *Let x and y be two points of G such that $D^k(x) \subset D^k(y)$, then x is joined to y , and $\Gamma^+(x) \cup \{x\} = \Gamma^+(y) \cup \{y\}$. In particular, if xy is an edge of G then x must be joined to every point inside $D_x(\|xy\|/2)$.*

Proof. Since $D^k(x) \subset D^k(y)$, the k nearest neighbours of y must all lie inside $D^k(x)$. If $y \notin D^k(x)$, then $D^k(y)$ contains $k+2$ points ($k+1$ in $D^k(x)$), which is impossible. Thus xy is an edge of G and the set of points (excluding x and y) in $D^k(x)$ is precisely the same as those in $D^k(y)$.

To prove the last part, suppose that z is a point in $D_x(\|xy\|/2)$. Then \overrightarrow{xz} must be an out edge, since $\|xz\| < \|xy\|$. Now, if \overrightarrow{zx} is not an out edge then $x \notin D^k(z)$, but $z \in D_x(\|xy\|/2)$, and so $D^k(z) \subset D_x(\|xy\|) \subset D^k(x)$. But this implies $xz \in G$ by the above. \square

We will now show that there is an absolute minimum distance between a point and a edge from a different component. As the main step to doing so, (and for most of the rest of this subsection) we show that there is a relative minimum distance between an edge of G and the distance of a point from a different component to that edge (as a function of the length of the edge). This result will be used both as the main part of that result of an absolute minimum distance, and later as part of the proof that with high probability edges in different components cannot cross. To this end we prove a fairly strong result and introduce a lot of the notation and set-up which we will meet again when proving that edges will not cross with high probability.

Lemma 5. *Suppose b_1 and b_2 are in a component X , with $b_1b_2 \in G$, $\|b_1b_2\| = \rho$ and $a \notin X$, then:*

$$d(a, b_1b_2) \geq \frac{1}{4\sqrt{6}}\rho > 0.102\rho \tag{1}$$

Proof. Suppose a , b_1 and b_2 are as above. We rescale and introduce Cartesian co-ordinates, fixing b_1 at $(0,0)$ and b_2 at $(1,0)$. Without loss of generality, $a^{(y)} \geq 0$ and $a^{(x)} \leq \frac{1}{2}$. We need to show that $d(a, b_1b_2) \geq \frac{1}{4\sqrt{6}}$. We write B_i

for $D_{b_i}(1)$, and note that $B_i \subset D^k(b_i)$ (as the edge $b_1b_2 \in G$). We may assume that $a \in B_1$, since otherwise $d(a, b_1b_2) \geq \frac{\sqrt{3}}{2}$ (as $a_1^{(x)} \leq 1/2$).

Since a is not joined to either b_i , Lemma 4 tells us that:

$$a \notin D_{b_1}(1/2) \cup D_{b_2}(1/2) \quad (2)$$

If $a^{(x)} < 0$, then, using (2), $d(a, b_1b_2) > 1/2$. Thus we may assume $0 < a^{(x)} \leq 1/2$, so that we have $d(a, b_1b_2) = a^{(y)}$.

Let w be the location $(\frac{1}{2}, \frac{1}{2\sqrt{3}})$, and let T be the triangle with vertices b_1 , b_2 and w (See figure 2).

Note that $b_1\widehat{b_2}w = b_2\widehat{b_1}w = \frac{\pi}{6}$, and so T intersects $D_{b_1}(1/2)$ and $D_{b_2}(1/2)$ at $(\frac{\sqrt{3}}{4}, \frac{1}{4})$ and $(1 - \frac{\sqrt{3}}{4}, \frac{1}{4})$ respectively. In particular, (2) tells that if $a \notin T$ then $d(a, b_1b_2) \geq \frac{1}{4}$.

Thus we may assume that $\overrightarrow{b_1a}$ and $\overrightarrow{b_2a}$ are out edges, and that:

$$a \in S = \left(T \cap \{p : p^{(x)} < \frac{1}{2}\} \right) \setminus D_{b_1}(1/2) \quad (3)$$

See Figure 2.

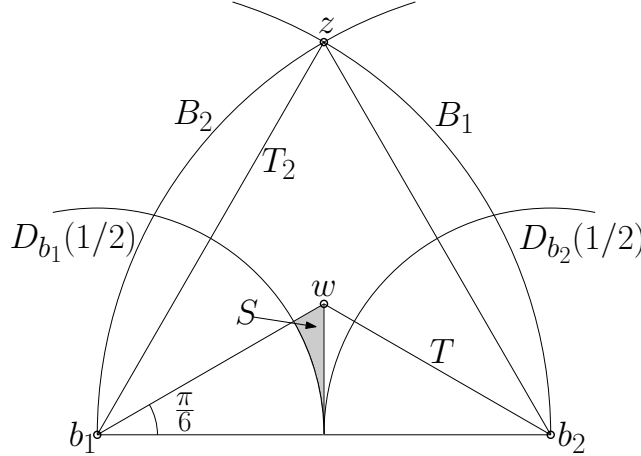


Figure 2: The region we are considering for a , shown with T and T_2 .

Define $r = \|ab_1\|$, and write A for the disk $D_a(r)$, so that $\Gamma^+(a) \subset D^k(a) \subset A$. Since $a \in S$, we have:

$$r \leq \|b_1w\| = \frac{1}{\sqrt{3}} \quad (4)$$

Let z be the location $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. Note that b_1 , b_2 and z form an equilateral triangle T_2 that contains T (See figure 2). Note that for any point in T_2 (and so, in particular, for every point in S), z is the closest point on $\partial(B_1 \cup B_2)$. Thus:

$$d(a, \partial(B_1 \cup B_2)) = \|az\| \geq \|wz\| = \frac{1}{\sqrt{3}} \quad (5)$$

Thus, putting (4) and (5) together, we have:

$$D^k(a) \subset A \subset B_1 \cup B_2 \quad (6)$$

Now, Lemma 4 tells us that we cannot have $\Gamma^+(a) \subset B_i$ for either i , and so $\Gamma^+(a)$ (and thus A) must contain points in both $B_1 \setminus B_2$ and $B_2 \setminus B_1$. We consider a point $p \in \Gamma^+(a) \cap (B_2 \setminus B_1)$. By definition, both b_2 and a must have an out edge to p , and thus, since a and b_2 are in different components, one of the following must hold:

1. p has no out edge to a .
2. p has no out edge to b_2 .

We will show that if a is too close to b_1b_2 , then A (and so $\Gamma^+(a)$) cannot contain a suitable point with either of these conditions holding. In particular, writing E for the ellipse $\{p : \|ap\| + \|b_2p\| \leq 1\}$, we show that if a is too close to b_1b_2 then $R := A \cap (B_2 \setminus B_1) \subset E \cap D_{b_2}(1/2)$, and that no point in $E \cap D_{b_2}(1/2)$ can satisfy either of the above conditions.

Lemma 6. *If $p \in E$ then \overrightarrow{pa} is an out edge. In particular, if $p \in E \cap D_{b_2}(1/2)$, then both \overrightarrow{pa} and $\overrightarrow{pb_2}$ are out edges.*

Proof. Suppose that $p \in E$ and \overrightarrow{pa} is not an out edge. We must have $a \notin D^k(p)$, and so $D^k(p) \subset B_2 \subset D^k(b_2)$ by the definition of E . Thus lemma 4 tells us that $\Gamma^+(p) \cup \{p\} = \Gamma^+(b_2) \cup \{b_2\}$. But $a \in \Gamma^+(b_2)$, and so $a \in \Gamma^+(p)$, and we have a contradiction.

The second part follows by applying Lemma 4. \square

We now identify a location, q , which is quite high up on ∂B_1 and must be inside $E \cap D_{b_2}(1/2)$. Lemma 6 tells us that R must contain a point further round ∂B_1 than q , or else a and b_2 are in the same component. This will force a itself to not be too close to b_1b_2 .

Lemma 7. *Let $q = (\frac{11}{12}, \frac{\sqrt{23}}{12})$. Then, so long as $a \in S$, $q \in E \cap D_{b_2}(1/2)$.*

Proof. We have that $\|qb_2\| = \sqrt{(\frac{1}{12})^2 + (\frac{\sqrt{23}}{12})^2} = \frac{1}{\sqrt{6}} < \frac{1}{2}$. Thus $q \in D_{b_2}(1/2)$, and moreover $q \in E$ if and only if $a \in D_q(1 - \frac{1}{\sqrt{6}})$.

Since S is contained within its complex hull, we will have $a \in D_q(1 - \frac{1}{\sqrt{6}})$ so long as the corners of S are contained within $D_q(1 - \frac{1}{\sqrt{6}})$. Now, S has three corners: $(\frac{1}{2}, 0)$, $(\frac{\sqrt{3}}{4}, \frac{1}{4})$ and $(\frac{1}{2}, \frac{1}{2\sqrt{3}})$, and by some simple calculations:

$$d(q, (\frac{1}{2}, \frac{1}{2\sqrt{3}})) < d(q, (\frac{1}{2}, 0)) < 1 - \frac{1}{\sqrt{6}}$$

And:

$$d(q, (\frac{\sqrt{3}}{4}, \frac{1}{4})) < 1 - \frac{1}{\sqrt{6}}$$

Thus all these locations are inside $D_q(1 - \frac{1}{\sqrt{6}})$, and we are done. \square

Note that $\|qb_1\| = 1$ and so $q \in \partial B_1$. Now, R must have its location furthest from b_2 on ∂B_1 (since $b_2 \in \partial B_1$ and $a \in B_1$), and so if R contains any location outside of $E \cap D_{b_2}(1/2)$ it must contain a location further up ∂B_1 than q .

Since R is symmetric about the line through a and b_1 , R could only contain a location above q if a is above the bisector of angle qb_1b_2 (denote this line L). Since we are assuming $a \in S$, we must have that $a^{(y)}$ (and so $d(a, b_1b_2)$) is at least the second co-ordinate of the intersection between $\partial D_{b_1}(1/2)$ and L .

Writing 2θ for $\widehat{qb_1b_2}$, we have that:

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2} = \left(1 - \frac{11/12}{\sqrt{(11/12)^2 + (\sqrt{23}/12)^2}}\right) / 2 = \frac{1}{24} \quad (7)$$

Now, a must be above the location which is $1/2$ along the line L from b_1 (since $a \notin D_{b_1}(1/2)$). Thus:

$$a^{(y)} \geq \frac{1}{2} \sin \theta = \frac{1}{2} \frac{1}{\sqrt{24}} = \frac{1}{4\sqrt{6}} \quad (8)$$

□

We want to bound the distance between a point and an edge in a different component independent of the length of the edge. We do this by applying Lemma 3 if the edge is short, and Lemma 5 if the edge is long:

Corollary 8. *With r as defined in Lemma 3, we have that if b_1 and b_2 are in a component X with $b_1b_2 \in G$, and $a \notin X$, then;*

$$d(a, b_1b_2) > \frac{r}{5} \quad (9)$$

Proof. Suppose b_1, b_2 and a are as above and let $\|b_1b_2\| = \rho$.

If $\rho \leq \frac{4\sqrt{6}}{5}r$: We may assume $\|ab_1\| \leq \|ab_2\|$. Then the perpendicular projection of a onto b_1b_2 is at most $\rho/2$ from b_1 . Thus, since ab_1 is not an edge of G , Lemma 3 tells us that $\|ab_1\| \geq r$ and so:

$$d(a, b_1b_2) \geq \sqrt{r^2 - (\rho/2)^2} \geq \sqrt{r^2 - \left(\frac{2\sqrt{6}}{5}r\right)^2} = \frac{r}{5} \quad (10)$$

If $\rho \geq \frac{4\sqrt{6}}{5}r$: By Lemma 5 we have that:

$$d(a, b_1b_2) \geq \frac{1}{4\sqrt{6}}\rho \geq \frac{r}{5} \quad (11)$$

□

Remark. *Lemma 5 can be improved, with substantial extra work, to show the distance between a and b_1b_2 is at least 0.1934ρ , which is best possible.*

3.2 Proof of Theorem 1 - Edges in different components cannot cross

In this section we will show:

Theorem 1 *If $k = c \log n$, then, for $c > 0.7102$, no two edges in different components inside G will cross with high probability.*

The value $c = 0.7102$ is strictly less than the current lower bound on the connectivity constant (i.e. $c = 0.7209$), and so edges in different components stop crossing before everything is connected.

The proof of Theorem 1 will split into three main parts. In the first we prove that for two such edges to cross, there must be a fairly specific set-up of points, more precisely it must look similar to the construction in Figure 1. In the second section we show that we can define two regions within this set-up, one of which has high density (containing at least k points and denoted H), and the other of which is empty (and denoted L). In the third section we bound the relative sizes of these two regions, and so achieve a bound on the likelihood of such a set-up occurring by using the following result of Balister, Bollobás, Sarkar and Walters [1], proved using simple properties of the Poisson process:

Lemma 9. *If X and Y are two regions of the plain, then:*

$$\mathbb{P}(\#X \geq k \text{ and } \#Y = 0) \leq \left(\frac{|X|}{|X| + |Y|} \right)^k$$

It is worth remarking that there will exist a constant c' such that if $k < c' \log n$ then with high probability we would have edges in different components crossing: We have a construction where we do have two edges in different components crossing (see Figure 1 in the introduction). Now, the construction has 5 dense regions, which we denote H_i ($i = 1, \dots, 5$), each of which contains m_i points, ($\sum_i m_i = 4k$) and a large empty regions, which we will denote L . If we have a region of the right shape with an area equal to the number of points in the construction (namely $4k$), then, writing p_n for the probability of the construction occurring in that region, we have:

$$\begin{aligned} p_n &> \prod_i^5 \left(\frac{|H_i|}{|L \cup H_i|} \right)^{m_i} \\ &> \min_{|H_i|} \left(\frac{|H_i|}{|L \cup H_i|} \right)^{4k} \\ &= n^{4c' \min_{|H_i|} \log \frac{|H_i|}{|L \cup H_i|}} \end{aligned} \tag{12}$$

when $k = c' \log n$. Now, by taking c' to be small enough, we can make the exponent of (12) arbitrarily close to 0, and so the probability of such a set-up occurring can be $O(n^{-\epsilon})$ for any $\epsilon > 0$. Since the region had an area of $O(\log n)$, we can fit $O(n/\log n)$ disjoint copies into S_n . Thus if we partition S_n into $O(n/\log n)$ regions in each of which the set-up could occur, it will occur in some of them with high probability, and so G will contain components with crossing edges with high probability.

3.2.1 The set-up of the points

To prove the result, we need to refer to several specific regions and locations within S_n , and so to make it easier to follow, all definitions and notation within this section are collated in the order that they appear in Appendix A, in addition to being defined inside this section.

Definition 2. We say that the ordered set of points: (a_1, a_2, b_1, b_2) forms a crossing pair if:

- The straight line segments a_1a_2 and b_1b_2 intersect and are both edges of the graph G ,
- the points a_1 and a_2 are in a different component from b_1 and b_2 ,
- $\|a_1a_2\| \leq \|b_1b_2\|$, $\|a_1b_1\| \leq \|a_1b_2\|$ and $d(a_1, b_1b_2) \leq d(a_2, b_1b_2)$.

Note that any four points that meet the first two conditions must also meet the third under a suitable identification of points, so that if two edges from different components cross then some four points must form a crossing pair.

We will use this definition of crossing pairs to determine exactly how a set-up with two edges from different components crossing must look. Given a crossing pair, we introduce Cartesian co-ordinates and rescale exactly as in Lemma 5 throughout this section (i.e. setting $b_1 = (0, 0)$, $b_2 = (1, 0)$, $a_1^{(x)} \leq 1/2$, $a_1^{(y)} \geq 0$ and $a_2^{(y)} \leq 0$). We now introduce some definitions of regions (dependent on a_1, a_2, b_1 and b_2), which we will use to pin point where these points can lie in relation to each other:

Definition 3. Let $r_i = \min\{\|a_i b_1\|, \|a_i b_2\|\}$ (so that $r_1 = \|a_1 b_1\|$) and define $A_i = D_{a_i}(r_i)$ and $B_i = D_{b_i}(\|b_1 b_2\|) = D_{b_i}(1)$ (See Figure 3).

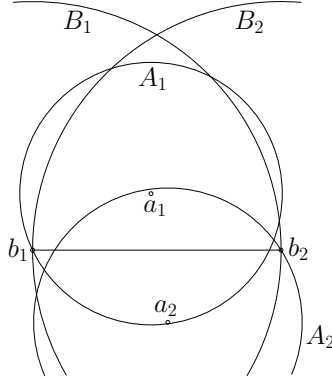


Figure 3: The regions A_1 , A_2 , B_1 and B_2 .

Definition 4. We write T for the isosceles triangle with vertices b_1 , b_2 and w where $w = (\frac{1}{2}, \frac{1}{2\sqrt{3}})$, and S_1 for the region $(T \cap \{q : q^{(x)} \leq 1/2\}) \setminus D_{b_1}(1/2)$ (This will turn out to be the region which can contain a_1 . See Figure 4).

Definition 5. We write T_2 for the equilateral triangle with vertices b_1 , b_2 and z , where $z = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$, and S_2 for the region $T_2 \cap A_1 \cap \{x : x \widehat{b_1 b_2} >$

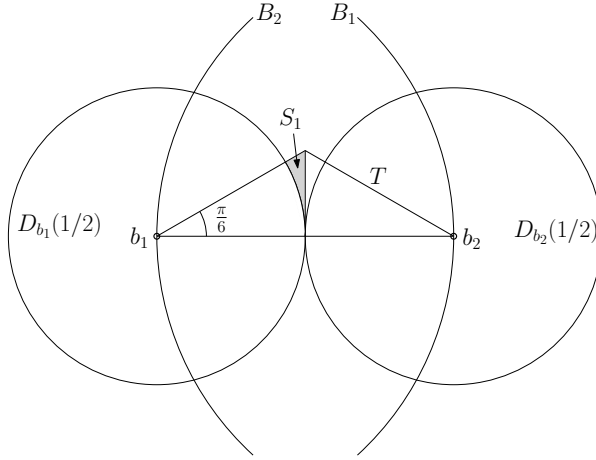


Figure 4: The shaded region is the region S_1 (which can contain a_1).

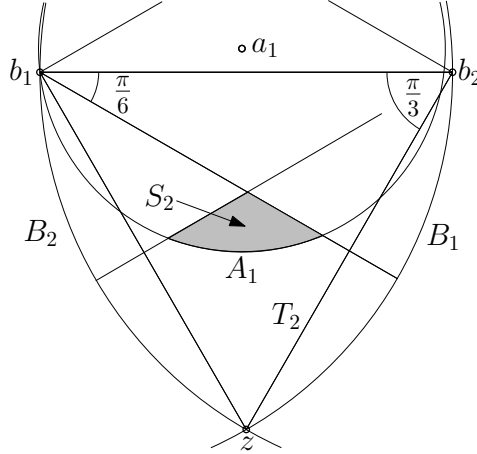


Figure 5: The shaded region is the region S_2 (which can contain a_2).

$\pi/6$ and $xb_2b_1 > \pi/6$ (This will turn out to be the region that can contain a_2 . See Figure 5).

Definition 6. For any set S , we define S^+ to be the part of S that lies above the x -axis (i.e. the line through b_1 and b_2), and S^- to be the part of S that lies below the x -axis.

To show that $a_1 \in S_1$ and $a_2 \in S_2$, (as well as later) we will need the following generalisation of Lemma 4 to pairs of points:

Lemma 10. Suppose w, x, y and z are any four points such that:

1. $D^k(w) \cup D^k(x) \subset D^k(y) \cup D^k(z)$,
2. $D^k(w) \cap D^k(x) \subset D^k(y) \cap D^k(z)$.

Then at least one of wy, wz, xy and xz is an edge of G .

Proof. Let $\#(D^k(w) \cap D^k(x)) = m$ and $\#(D^k(y) \cap D^k(z)) = \mu$. Then, by condition 2, $m \leq \mu$. However, $\#(D^k(w) \cup D^k(x)) = 2k + 2 - m$ and $\#(D^k(y) \cup D^k(z)) = 2k + 2 - \mu$, and so condition 1 implies $2k + 2 - m \leq 2k + 2 - \mu$ and thus $m \geq \mu$. Putting these together, we must have $m = \mu$.

This tells us that $\#(D^k(w) \cup D^k(x)) = \#(D^k(y) \cup D^k(z))$, and so, by condition 1, we have $\Gamma^+(w) \cup \Gamma^+(x) \cup \{w, x\} = \Gamma^+(y) \cup \Gamma^+(z) \cup \{y, z\}$. In particular $w, x \in \Gamma^+(y) \cup \Gamma^+(z)$ and $y, z \in \Gamma^+(w) \cup \Gamma^+(x)$, and so each of w and x receives an out-edge from at least one of y and z and each of y and z receives an out-edge from at least one of w and x . We may assume by symmetry that \overrightarrow{wy} is an out-edge.

Now, if wy were not an edge of G , then \overrightarrow{zw} must be an out-edge (since one of \overrightarrow{yw} and \overrightarrow{zw} must be). Similarly, if zw is not an edge of G either, then \overrightarrow{xz} must be an out edge. Continuing, we find that either one of wy, wz, xy and xz is an edge of G , or all of $\overrightarrow{wy}, \overrightarrow{zw}, \overrightarrow{xz}, \overrightarrow{yx}$ are out-edges, but none are in-edges. This would imply:

$$\|wy\| < \|zw\| < \|xz\| < \|yx\| < \|wy\|,$$

which is impossible. \square

We now finish this sub-section by showing that $a_1 \in S_1$ and $a_2 \in S_2$, and proving some other basic facts about crossing pairs:

Lemma 11. *Suppose (a_1, a_2, b_1, b_2) forms a crossing pair, then:*

1. a_1a_2 must be the shortest edge in the convex quadrilateral $a_1a_2b_1b_2$,
2. we must have $0 < a_1^{(x)}, a_2^{(x)} < 1$, and $B_i \subset D^k(b_i)$ and $\Gamma^+(a_i) \subset A_i$ for $i = 1, 2$,
3. $a_1 \in S_1$,
4. for any point $p \in T_2$ with $b_1, b_2 \notin D^k(p)$, if either of $b_1\widehat{b_2}p \leq \pi/6$ or $b_2\widehat{b_1}p \leq \pi/6$ then $D^k(p) \subset B_1 \cup B_2$,
5. $a_2 \in S_2$.

Proof. 1. Since a_1a_2 and b_1b_2 intersect, the four points must form a convex quadrilateral with a_1a_2 and b_1b_2 as the diagonals.

Suppose a_1b_1 is shorter than a_1a_2 (and so also shorter than b_1b_2), then $a_1 \in D^k(b_1)$ as b_2 is, and $b_1 \in D^k(a_1)$ as a_2 is. Thus a_1b_1 is an edge in G , contradicting (a_1, a_2, b_1, b_2) being a crossing pair. Similarly, a_ib_j cannot be shorter than both a_1a_2 for any i and j .

2. We know that $b_1b_2 \in G$, and thus $B_i \subset D^k(b_i)$, and know already that $a_1^{(x)} \leq \frac{1}{2}$.

Suppose that $a_1^{(x)} \leq 0$. Since a_1a_2 and b_1b_2 intersect, we must have $a_2^{(x)} > 0$. But then $\|b_1a_2\| < \|a_1a_2\|$, contradicting part 1. Thus $a_1^{(x)} > 0$. The same argument shows that $a_2^{(x)} > 0$ and $a_2^{(x)} < 1$.

By the above, and using $\|a_1a_2\| \leq \|b_1b_2\| = 1$ as well as $d(a_1, b_1b_2) \leq d(a_2, b_1b_2)$, we have that $0 \leq a_1^{(y)} = d(a_1, b_1b_2) \leq \frac{1}{2}$. We also know that $0 < a_1^{(x)} \leq \frac{1}{2}$, and so $\|a_1b_1\| \leq \frac{1}{\sqrt{2}}$, and in particular $a_1 \in B_1$.

Thus $\overrightarrow{b_1 a_1}$ is an out edge, and so $b_1 \notin \Gamma^+(a_1)$ as $a_1 b_1$ is not an edge of G . This implies that $b_2 \notin \Gamma^+(a_1)$ as $a_1^{(x)} \leq \frac{1}{2}$. Thus $D^k(a_1) \subset A_1$.

Since neither b_1 nor b_2 are in A_1 and $0 < a_1^{(x)} \leq \frac{1}{2}$, we must have $(\partial A_1)^- \subset B_1 \cap B_2$. Thus $\overrightarrow{D^k(a_1)^-} \subset A^- \subset B_1 \cap B_2$, and so $a_2 \in B_1 \cap B_2$ implying that $\overrightarrow{b_1 a_2}$ and $\overrightarrow{b_2 a_2}$ are both out edges. Thus neither b_1 nor b_2 are in $\Gamma^+(a_2)$, so $D^k(a_2) \subset A_2$.

3. We must have $2d(a_1, b_1 b_2) \leq \|a_1 a_2\| \leq \|a_1 b_1\|$, since $0 < a_1^{(x)}, a_2^{(x)} < 1$ and $a_1 a_2$ is the shortest edge in our quadrilateral, and so in particular:

$$d(a_1, b_1 b_2) \leq \frac{1}{2} \|a_1 b_1\|$$

Thus, using $\|a_1 b_1\| \leq \|a_1 b_2\|$:

$$a_1 \widehat{b_2 b_1} \leq a_1 \widehat{b_1 b_2} \leq \sin^{-1}\left(\frac{1}{2}\right) = \pi/6 \quad (13)$$

This is exactly the region T , and since $a_1^{(x)} \leq 1/2$ and $a_1 \notin D_{b_1}(1/2)$ (by Lemma 4), we have:

$$a_1 \in \left(T \cap \{q : q^{(x)} \leq 1/2\}\right) \setminus D_{b_1}(1/2) = S_1$$

4. Let $p \in T_2$ be such that $b_1, b_2 \notin D^k(p)$. Note that z is the closest location to p in $\partial(B_1 \cup B_2)$ (since $p \in T_2$), and so in particular $D_p(\|pz\|) \subset B_1 \cup B_2$. Thus it suffices to show that $z \notin D^k(p)$.

If $b_1 \widehat{p b_2} \leq \pi/6$, then $\|b_1 p\| \leq \|p z\|$ since the line $\{q : b_1 \widehat{b_2 q} = \pi/6\}$ bisects $b_1 \widehat{b_2 z}$. Thus in particular, $z \notin D^k(p)$ since $b_1 \notin D^k(p)$.

Similarly, if $b_2 \widehat{p b_1} \leq \pi/6$ then $z \notin D^k(p)$.

5. Noting that the a_i and b_i fulfil condition 2 of Lemma 10 (with the identification, in the notation of Lemma 10, of $a_1 = w$, $a_2 = x$, $b_1 = y$ and $b_2 = z$), and so, since the a_i and b_i are in different components, Lemma 10 implies that $A_1 \cup A_2 \not\subset B_1 \cup B_2$. Thus at least one of a_1 and a_2 must be closer to a point outside of $B_1 \cup B_2$ than it is to b_1 and b_2 . This cannot be a_1 by parts 3 and 4. Thus a_2 is closer to a point outside of $B_1 \cup B_2$ than it is to b_1 or b_2 .

Since $a_1 a_2$ is the shortest edge in both triangles $a_1 a_2 b_1$ and $a_1 a_2 b_2$, we have $a_1 \widehat{b_i a_2} \leq \pi/3$ for $i = 1, 2$, and so $a_2 \in T_2$. Thus by part 4, $a_2 \widehat{b_1 b_2} > \pi/6$ and $a_2 \widehat{b_2 b_1} > \pi/6$. We also know that $a_2 \in A_1$ as $a_1 a_2 \in G$, whence:

$$a_2 \in T_2 \cap A_1 \cap \{x : x \widehat{b_1 b_2} > \pi/6 \text{ and } x \widehat{b_2 b_1} > \pi/6\} = S_2$$

□

3.2.2 The dense and empty regions

We want to define our regions of high and low density, but first need some more basic regions that they will be built from. We define:

- R_i to be $D^k(a_1) \cap (B_i \setminus B_j)$ where $i \neq j$,
- E_i to be the ellipse defined by the equation $\|a_1x\| + \|b_ix\| \leq 1$ (This has its centre half way between a_1 and b_i , major axis running along the line a_1b_i with radius $1/2$, and minor axis of radius $\frac{\sqrt{1-r_i^2}}{2}$),
- F_i to be the ellipse defined by the equation $\|a_2x\| + \|b_ix\| \leq 1$,
- M to be $D^k(a_1) \cap D^k(a_2)$.

We can now define all our regions of high and low density (and will prove they are such shortly). All these regions are shown in Figure 6. The empty regions are:

- $L_1 = (D^k(a_1)^+ \cap E_1 \cap D_{b_1}(1/2)) \setminus M$
- $L_2 = (D^k(a_1)^+ \cap E_2 \cap D_{b_2}(1/2)) \setminus M$
- $L_3 = M^+ \cap (D_{b_1}(1/2) \cup D_{b_2}(1/2))$
- $L_4 = T_2 \cap D^k(a_2) \cap \{x : x\hat{b}_1b_2 \leq \pi/6 \text{ or } x\hat{b}_2b_1 \leq \pi/6\}$
- $L_5 = (D^k(a_2)^- \cap F_1 \cap D_{b_1}(1/2)) \setminus T_2$
- $L_6 = (D^k(a_2)^- \cap F_2 \cap D_{b_2}(1/2)) \setminus T_2$.

The high density regions are:

- $H_1 = R_1 \setminus L_1$
- $H_2 = R_2 \setminus L_2$
- $H_3 = A_2^- \setminus (B_1 \cup B_2)$
- $H_4 = M^+ \setminus L_3$.
- $H_5 = S_2$.

And we write:

$$H = \bigcup_{i=1}^5 H_i \tag{14}$$

$$L = \bigcup_{i=1}^6 L_i \tag{15}$$

See Figure 6 for an illustration of this.

We want to show that L is empty, and that H contains at least k points. To do this we will first show that $H \cup L$ contains at least k points and then show that $\#L = 0$.

Lemma 12. *With the regions as defined above, we have $\#(H \cup L) > k$.*

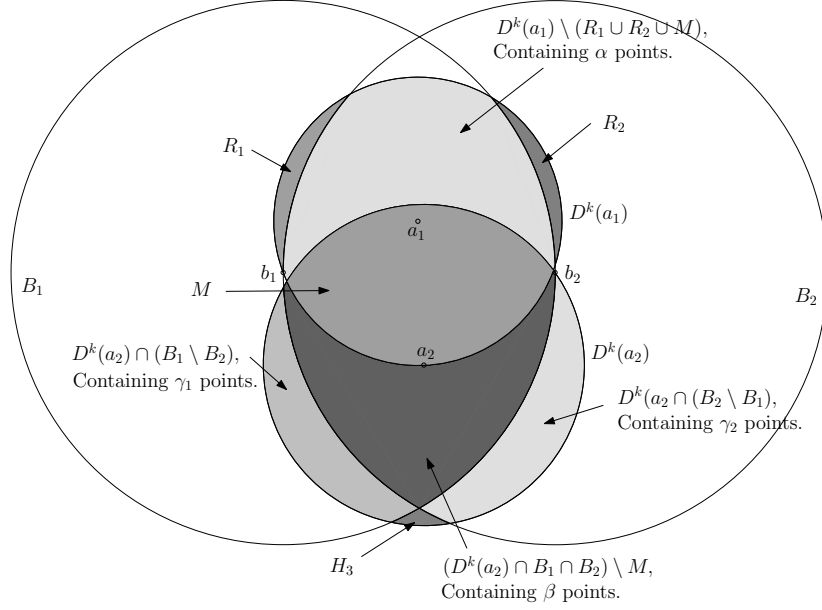


Figure 7: The regions we are considering, with their number of points.

Thus, by using (16), (21), (22) and finally (18) we get:

$$\begin{aligned}
\#(H \cup L) &\geq \#H_3 + \#M + \#R_1 + \#R_2 \\
&\geq \#H_3 + \#M + (\beta + \gamma_2 + 1) + (\beta + \gamma_1 + 1) \\
&= (\#H_3 + \#M + \beta + \gamma_1 + \gamma_2) + (\beta + 2) \\
&= k + \beta + 3 \\
&> k
\end{aligned}$$

□

We next show that for each i , $\#L_i = 0$.

Lemma 13. $\#L_1 = \#L_2 = \#L_5 = \#L_6 = 0$.

Proof. Lemma 6 tells us that any point in L_1 has an out edge to both a_1 and b_1 , but L_1 is contained inside both $D^k(a_1)$ and $D^k(b_1)$, and thus must be empty. Similarly for L_2 , L_5 and L_6 . □

The cases for L_3 and L_4 require slightly more work and are dealt with separately.

Lemma 14. $\#L_4 = 0$

Proof. Note that L_4 is contained in the polygon, P , with corners (moving around its perimeter clockwise) at b_1 , b_2 , $u^- = (\frac{3}{4}, -\frac{\sqrt{3}}{4})$, $w^- = (\frac{1}{2}, -\frac{1}{2\sqrt{3}})$ and $v^- = (\frac{1}{4}, -\frac{\sqrt{3}}{4})$. We will show that the left half of this region (namely the convex polygon P^l , with corners b_1 , $(\frac{1}{2}, 0)$, w^- and u^-) is contained within F_1 , and then use Lemma 6 to show that we can have no points in $L_4 \cap P^l$. To do this it is convenient to first bound S_2 into a convex polygon:

By Lemma 5, $a_1^{(y)} \geq 0.102$, and thus the minimal possible y co-ordinate of a point $q \in M^-$ (and so for a_2) can be no less than the minimum when taking a_1 to be at $(1/2, 0.102)$ and $D^k(a_1) = A_1$. This bounds $q^{(y)}$ (and in particular $a_2^{(y)}$) below by:

$$q^{(y)} \geq 0.102 - \sqrt{(1/2)^2 + 0.102^2} > v^{-(y)} = -\frac{\sqrt{3}}{4}$$

Thus S_2 is contained in the triangle T_{a_2} , with corners u^- , v^- and w^- .

By convexity, to check that $P^l \subset F_1$ it is enough to check that for every corner of P^l and every corner of T_{a_2} (labelling these corners by p_i and t_j respectively) the equation

$$\|b_1 p_i\| + \|p_i t_j\| \leq 1$$

holds. This is the case (calculations omitted), and so $P^l \subset F_1$.

Lemma 6 then tells us that any point in $L_4 \cap P^l$ must have an out-edge to both b_1 and a_2 , but $P^l \subset B_1$ and $L_4 \subset D^k(a_2)$, so any point in $L_4 \cap P^l$ would then be joined to both b_1 and a_2 in G , and so no such point can exist. Similarly, defining P^r to be the right half of P , $L_4 \cap P^r$ must be empty, and so $\#L_4 = 0$. \square

Lemma 15. *The region $L_3 \cap \{p : p^{(x)} < \frac{1}{2}\} \subset E_1$ and $L_3 \cap \{p : p^{(y)} \geq \frac{1}{2}\} \subset E_2$, and so in particular $\#L_3 = 0$.*

Proof. We show that L_3 is contained in the polygon Q with corners (moving around its perimeter clockwise) at b_1 , $u^+ = (\frac{1}{6}, \frac{1}{2\sqrt{3}})$, $v^+ = (\frac{5}{6}, \frac{1}{2\sqrt{3}})$ and b_2 . The proof will then follow as in Lemma 14; we show that the left and right halves of Q are contained in E_1 and E_2 respectively, and use this to rule out any points in L_3 .

Writing z^+ for the location $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, we have that $b_1 \widehat{b_2} z^+ = b_2 \widehat{b_1} z^+ = \frac{\pi}{3}$. Now, $L_3 \subset A_2^+$ (by Lemma 11 part 2), and $a_2 \widehat{b_i} z^+ \geq \frac{\pi}{2}$ (by Lemma 11 part 5), and thus, since $a_2 \widehat{b_i} b_j \geq \frac{\pi}{6}$ ($i \neq j$), it follows that L_3 is contained in the triangle with vertices b_1 , b_2 and $z^+ = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ (as $L_3 \subset A_2^+$). Now, u^+ and v^+ lie on the lines $b_1 z^+$ and $b_2 z^+$ respectively, and so we just need to show that L_3 can't come too high up inside this triangle: By Lemma 11 part 5, $a_2^{(y)} \leq -\frac{1}{2\sqrt{3}}$, and thus the maximal possible y co-ordinate of a point $q \in M^+$ can be no more than the maximum when taking a_2 to be at $(1/2, -\frac{1}{2\sqrt{3}})$ and $D^k(a_2) = A_2$. This bounds $q^{(y)}$ above by:

$$q^{(y)} \leq \frac{1}{2\sqrt{3}}$$

Thus every point in M^+ , and hence every point in L_3 , is inside Q .

By writing Q^l for the left half of Q , q_i for the corners of Q^l and noting that S_1 (and hence a_1) is contained in the convex polygon T_{a_1} with corners t_j at $(\frac{1}{2}, 0)$, $(\frac{\sqrt{3}}{4}, \frac{1}{4})$, w and $(1 - \frac{\sqrt{3}}{4}, \frac{1}{4})$, it follows by convexity that since all of the equations $\|b_1 q_i\| + \|q_i t_j\| \leq 1$ hold, $Q^l \subset E_1$. Lemma 6 and the definition of L_3 then tell us we can have no points inside $L_3 \cap Q^l$. Similarly we can have no points in $L_3 \cap Q^r$, where Q^r is the right half of Q , and so $\#L_3 = 0$. \square

Putting Lemmas 12–15 together we have:

Lemma 16. $\#H \geq k$ and $\#L = 0$.

□

3.2.3 Bounding the relative areas of H and L and the proof of Theorem 1

We define ρ_1 and ρ_2 to be the radius of $D^k(a_1)$ and $D^k(a_2)$ respectively and now move on to bound the relative areas of H and $H \cup L$. However, the regions defined above are quite complicated in shape, and so computing the relative areas, even for particular positions of a_1 and a_2 and given values of ρ_1 and ρ_2 , involves some complicated integrals. Moreover, we need to bound the relative areas over all possible positions of a_1 and a_2 and all allowable values of ρ_1 and ρ_2 . To obtain a bound we will thus break things down into finite cases as follows:

We first tile S_n with small squares and then consider the possible pairs of tiles which can contain a_1 and a_2 . For each such pair, we will bound $|H|$ above and $|L|$ below, and thus bound H above and $|L|$ below absolutely over all positions of a_1 and a_2 .

Practically, this requires the use of a computer, but will still be completely rigorous.

To make the calculations as simple as possible, we wish to reduce the number of variables we have to maximise and minimise over. In light of this we split L and H into two parts, each of whose size will be dependent on the position of only one of a_1 and a_2 (we will show this on a case by case basis later); namely L splits into $L^+ = L_1 \cup L_2 \cup L_3$ and $L^- = L_4 \cup L_5 \cup L_6$ and H splits into $H_1 \cup H_2 \cup S_2$ and $H_3 \cup H_4$. Further, it is easy to see that for any fixed positions of a_1 and a_2 , the area of any part of H will be maximised by maximising ρ_1 and ρ_2 , and that the area of any part of L will be minimised by minimising ρ_1 and ρ_2 . Thus, for each of the given parts of H or L above, we need only to bound the integral over the position of one of a_1 and a_2 and nothing else.

Our exact method is as follows: We tile S_n with small squares of side length s , which are aligned with the edge b_1b_2 , i.e. b_1b_2 will run along the edges of all the square it touches, and both b_1 and b_2 will be on the corners of squares (to prove our bound, we will use a square side length of $s = 0.001\|b_1b_2\|$). Whilst bounding an area dependent on the position of a_i , and given some small square X with centre x , we define σ_i^X and ρ_i^X to be the minimum and maximum values of ρ_i over all possible positions of a_i within X . We can then bound the area of the relevant part of H above by simply counting every square that could be within the part of H that contains any location within ρ_i^X of any location in X , and bound the area of the relevant part of L below by counting only squares that are entirely within that part of L and are entirely within σ_i^X of every location within X . In fact, it suffices to count every square that has its centre within $\rho_i^X + s\sqrt{2}$ of x for the bound on H , and only squares that have their centres within $\sigma_i^X - s\sqrt{2}$ of x for the bound on L , since this can only weaken the bounds obtained. We can then bound the areas of the relevant parts of H and L above and below respectively by taking the maximum and minimum of these sums over every square that could possibly contain a_i .

Since the regions we are using are often dependent on the ellipses E_i and F_i , and these are dependent on the position of a_1 and a_2 , it is useful to define:

$$E_i^X = \{q \in S_n : \max_{a \in X} \|b_i q\| + \|a q\| \leq 1\}$$

Similarly we define F_i^X when $a_2 \in X$. Thus E_i^X is the intersection of the $E_1(a_1)$ over all possible positions of a_1 within X . It is worth noting that when a region in L depends on an ellipse, it is contained within the ellipse, and when a region in H depends on an ellipse, it is outside the ellipse, so we will always want to use the intersection of the possible ellipses to bound our area, rather than a union. Note also that any small square Y , with centre y , such that $\|b_i y\| + \|xy\| \leq 1 - \frac{3\sqrt{2}}{2}s$, will be entirely contained within E_i^X .

Lemma 17. $|L^+| > 0.3411$

Proof. Note that:

$$L^+ = L_1 \cup L_2 \cup L_3 \quad (23)$$

$$= (D^k(a_1)^+ \cap E_1 \cap D_{b_1}(\frac{1}{2})) \cup (D^k(a_1)^+ \cap E_2 \cap D_{b_2}(\frac{1}{2})) \quad (24)$$

$$= D^k(a_1)^+ \cap [(E_1 \cap D_{b_1}(\frac{1}{2})) \cup (E_2 \cap D_{b_2}(\frac{1}{2}))] \quad (25)$$

Where (24) follows from (23) by Lemma 15. Thus $|L^+|$ does not depend on a_2 , and so is a function of the position of a_1 and ρ_1 only.

We know that $D^k(a_1)$ must contain a_2 as well as at least one point in H_1 (i.e. in R_1 and outside of $E_1 \cap D_{b_1}(1/2)$) and at least one point in H_2 (i.e. in R_2 and outside of $E_2 \cap D_{b_2}(1/2)$). Call the closest locations to a_1 in H_1 and H_2 , h_1 and h_2 respectively, and note that they are dependent only on the position of a_1 .

Now, given that a_1 is in some small square X with centre x , we set h_1^X to be the lower down (on $\partial B_2 = \partial D_{b_2}(1)$) of the two location $\partial B_2 \cap \partial D_{b_1}(1/2)$ and the location q on ∂B_2 for which $\|b_1 q\| + \|xq\| = 1 - \frac{\sqrt{2}}{2}s$, and similarly define h_2^X . Thus h_1^X (correspondingly h_2^X) is at least as far down ∂B_2 (correspondingly ∂B_1) as h_1 (or h_2) for any position of a_1 within X . Thus we define:

$$\rho = \max\{\|xh_1^X\|, \|xh_2^X\|, \|xa_2\|\} - \frac{\sqrt{2}}{2}s \leq \sigma_1^X$$

Then a small square Y with centre y will be entirely within L^+ regardless of where in X a_1 lies, so long as:

- Y is entirely above the line $b_1 b_2$,
- $\|yx\| \leq \rho - s\sqrt{2}$ (note that $s\frac{\sqrt{2}}{2}$ is subtracted twice from ρ_{\min}^X to account for the possible locations of points within both of the squares X and Y) and finally,
- every point in Y is inside both $D_{b_1}(1/2)$ and E_1^X or every point in Y is inside both $D_{b_2}(1/2)$ and E_2^X .

See Figure 8.

Performing our numerical integration on a computer then gives us $|L^+| > 0.3411 \dots$ with the minimum achieved when a_1 was in either of the squares with centres at $(0.4995, 0.1895)$ and $(0.5005, 0.1895)$. \square

Lemma 18. $|L^-| > 0.3564$

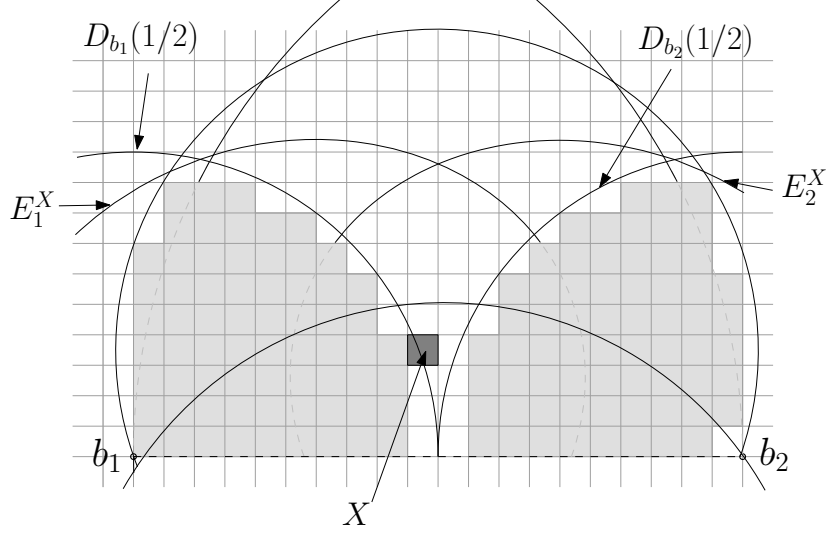


Figure 8: An incidence of the squares that will be counted as being in L^+ .

Proof. Note that:

$$L^- = L_4 \cup L_5 \cup L_6$$

None of the definitions of L_4 , L_5 or L_6 are dependent of the position of a_1 or the value of ρ_1 , although the region where we can place a_2 (i.e. the region S_2) is dependent on a_1 . From Lemma 5 we know that we cannot have a_1 as low as the point $(\frac{1}{2}, \frac{1}{4\sqrt{6}})$, and so, using Lemma 11 we may assume a_1 is at $(\frac{1}{2}, \frac{1}{4\sqrt{6}})$ and ρ_1 is maximal when determining if a small square contains a possible location in S_2 .

Given that a_2 is in some small square X with centre x , we can define:

$$\sigma = \max\{\|xa_1\|, \|xz\|\} - \frac{\sqrt{2}}{2}s \leq \sigma_2^X$$

Then a small square Y , with centre y , will be entirely within L^- regardless of where in X a_2 lies, so long as:

- Y is entirely below the line b_1b_2 ,
- $\|yx\| \leq \sigma - s\sqrt{2}$,
- every point $q \in Y$:
 1. is inside both $D_{b_1}(1/2)$ and F_1^X ,
 2. or is inside $D_{b_2}(1/2)$ and F_2^X ,
 3. or has $q \in T_2$ and either $b_1\widehat{b_2q} < \frac{\pi}{6}$ or $b_2\widehat{b_1q} < \frac{\pi}{6}$.

Computer calculations then gives $|L^-| > 0.3564\dots$ with a minimum value achieved when a_2 was in either of the squares with centres at $(0.4995, -0.3825)$ and $(0.5005, -0.3825)$. \square

Lemma 19. $|H_1 \cup H_2 \cup S_2| < 0.1300$.

Proof. The areas of H_1 , H_2 and S_2 all depend only on the position of a_1 and the value of ρ_1 , and thus to bound their union above we may assume that a_2 is located at $(\frac{1}{2}, -\frac{1}{2\sqrt{3}})$ and ρ_2 is maximal, as in lemma 17. We know also that $D^k(a_1) \subset A_1$, so that neither b_1 nor b_2 are within ρ_1 of a_1 .

Given that a_1 is in some small square X with centre x , the above tells us that, defining:

$$\tau = \min\{\|b_1x\|, \|b_2x\|\} + \frac{\sqrt{2}}{2}s \geq \rho_1^X$$

Then a small square Y , with centre y , can have some part of itself in H_1 , H_2 or S_2 only if:

- $\|yx\| \leq \tau + s\sqrt{2}$ and
- we have one of the following:
 1. Any location in Y is inside R_1 and outside of either E_1^X or $D_{b_1}(1/2)$ (Y contains a location in H_1)
 2. Any location in Y is inside R_2 and outside of either E_2^X or $D_{b_2}(1/2)$ (Y contains a location in H_2)
 3. Any location $q \in Y$ has $b_1\widehat{b_2}q \geq \frac{\pi}{6}$ and $b_2\widehat{b_1}q \geq \frac{\pi}{6}$ (Y contains a location in S_2).

Computer calculations then give $|H_1 \cup H_2 \cup S_2| < 0.1299\dots$ with a maximum achieved when a_1 was in the square with centre at $(0.4995, 0.2885)$. \square

Lemma 20. $|H_3 \cup H_4| < 0.0958$.

Proof. The areas of H_3 and H_4 depend only on the position of a_2 and the value of ρ_2 , and that when calculating whether a small square could contain a location in S_2 , we may assume that a_1 is at $(\frac{1}{2}, \frac{1}{4\sqrt{6}})$ and ρ_1 is maximal, as in Lemma 18.

Given that a_2 is in some small square X with centre x , the above tells us that, defining:

$$v = \min\{\|b_1x\|, \|b_2x\|\} + \frac{\sqrt{2}}{2}s \geq \rho_2^X$$

Then a small square Y with centre y can have some part of itself in H_3 or H_4 only if:

- $\|yx\| \leq v + s\sqrt{2}$ and
- either of the following holds:
 1. Any location in Y is outside $B_1 \cup B_2$ (Y contains a location in H_3)
 2. Any location in Y is above the line b_1b_2 and is outside $D_{b_1}(1/2) \cup D_{b_2}(1/2)$ (Y contains a location in H_4)

Our computer calculations gives us that $|H_3 \cup H_4| < 0.0957\dots$ with a maximum achieved when a_2 was in the square with centre at $(0.4995, -0.4335)$. \square

We can use Lemmas 17-20 to bound the ratio $\frac{|H|}{|H \cup L|}$:

Lemma 21. $\frac{|H|}{|H \cup L|} < 0.2446$.

Proof. Note that since H and L are disjoint, $\frac{|H|}{|H \cup L|} = \frac{|H|}{|H| + |L|}$, which is strictly increasing in $|H|$ and decreasing in $|L|$. Thus, by using Lemmas 17-20 we have:

$$\begin{aligned} \frac{|H|}{|H \cup L|} &< \frac{0.1300 + 0.0958}{0.1300 + 0.0958 + 0.3411 + 0.3564} \\ &< 0.2446 \end{aligned}$$

□

Using all of the above, we can finally prove Theorem 1:

Proof of Theorem 1. We pick six points $a_1, a_2, b_1, b_2, a_1^{(k)}$ and $a_2^{(k)}$, and write Z for the event that a_1, a_2, b_1 and b_2 form a crossing pair, and that $a_1^{(k)}$ and $a_2^{(k)}$ are the k^{th} nearest neighbours of a_1 and a_2 respectively.

When Z occurs, these six points define the regions H and L , and so for any given six tuple of points, Lemmas 16 and 21 tell us:

$$\begin{aligned} \mathbb{P}(Z) &\leq \left(\frac{|H|}{|H \cup L|} \right)^k \\ &< n^{c \log 0.2446} \end{aligned} \tag{26}$$

Now, there are $O(n)$ choices for a_1 , and once this has been chosen there are only $O(\log n)$ choices for each of $a_2, b_1, b_2, a_1^{(k)}$ and $a_2^{(k)}$ (since all five have either an out edge to or from a_1 (except for $a_2^{(k)}$ which must have an out edge from a_2), and so must be within $O(\sqrt{\log n})$ of a_1 by Lemma 3). Thus there are $O(n \log^5 n)$ choices for our system, and so, with high probability, no two edges in different components cross so long as:

$$c \log 0.2446 < -1$$

or equivalently:

$$c > 0.7102$$

3.3 There can only be one large component

We use Lemma 8 and Theorem 1 to get a bound on the absolute distance between any two edges in different components:

Corollary 22. *If $k = c \log n$, and $c > 0.7102$, then with high probability the minimal distance between two edges in different components is at least $r/5$, where r is as given in Lemma 3.*

Proof. Since $c > 0.7102$ we may assume, by Theorem 1, that no two edges in different components cross. Thus the minimal distance between two such edges will be at the end point of one of them. Corollary 8 then gives us the result. □

Using the above, we now meet all of the conditions for Lemma 12 of [1] so long as $k > 0.7102 \log n$, except that now the minimal distance between edges in different components is $r/5$ instead of $r/2$, however this requires only trivial changes in the proof, and so we gain:

Proposition 23. *For fixed $c > 0.7102$, if $k > c \log n$, then there exists a constant c' such that the probability that $G_{n, \lfloor c \log n \rfloor}$ contains two components of (Euclidean) diameter at least $c' \sqrt{\log n}$ tends to zero as $n \rightarrow \infty$.*

□

4 The main result

4.1 Approach and simple bound

Using the results from the previous section we can now proceed to gain an upper bound for the threshold for connectivity by ruling out the chance of having a small component.

We wish to prove a good bound on the critical constant c such that if $k > c \log n$ then $\mathbb{P}(G_{n,k} \text{ disconnected}) \rightarrow 0$ as $n \rightarrow \infty$. Proposition 23 tells us that if G is not connected, and $k > 0.7102 \log n$, then we may assume that there is a small component somewhere. In the next section we will show that such a small component will not exist with high probability for $c > 0.9684$, but first illustrate a simpler proof that works for $c > 1.0293$ to give the general approach. This proof is similar to the first part of Theorem 15 of [1]. We start by introducing some notation:

Definition 7. *Let d be $\max\{c', 4\sqrt{c_+/\pi}, \frac{1}{4\sqrt{c_-/\pi}}, 1\}$, (where c_+ and c_- are the constants from Lemma 3, and c' is the constant given by Proposition 23).*

Given four points, a, b, x_l and x_r in S_n , we define $\rho = \|ab\|$ and, writing $D_x^l(y)$ and $D_x^r(y)$ for the left and right half-disks of radius y centred on x , we define the regions:

- $C = (D_{x_l}^l(\rho) \cup D_{x_r}^r(\rho)) \cap S_n$,
- $A = (D_a(\rho) \setminus (D_b(\rho) \cup C)) \cap S_n$, and
- $B = (D_b(\rho) \setminus (D_a(\rho) \cup C)) \cap S_n$.

See Figure 9 for an illustration of these regions.

We say that a, b, x_l and x_r form a component set-up if:

1. *The points b, x_l and x_r are all within $d\sqrt{\log n}$ of a ,*
2. *$\#C = 0$,*
3. *and at least one of $\#A \geq k$ and $\#B \geq k$ holds.*

Lemma 24. *If there is a component, X , of diameter at most $d\sqrt{\log n}$ in G , then with high probability some four points form a component set-up.*

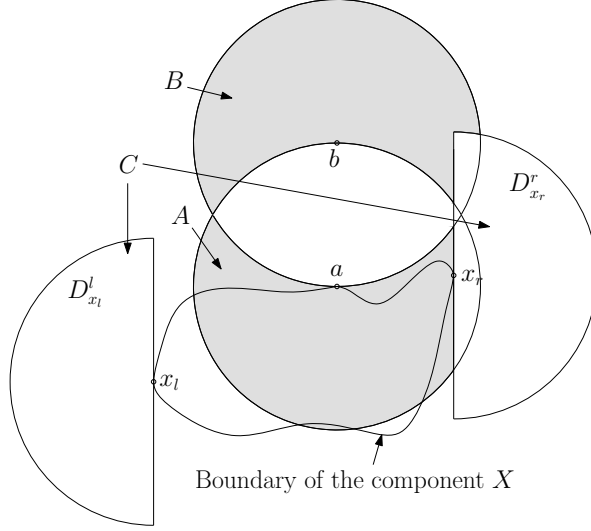


Figure 9: The set up of the points a, b, x_l and x_r and the regions they define.

Proof. Let $a \in X$ and $b \notin X$ be such that they minimise $\|ab\|$ over all such pairs. Let x_l be the left most point in the component X and x_r the right most point. We show that these four points form a component set-up with high probability.

Since $\text{diam}(X) \leq d\sqrt{\log n}$, x_l and x_r are within $d\sqrt{\log n}$ of a , and Lemma 3 tell us that b is within $d\sqrt{\log n}$ of a with high probability, so Condition 1 holds with high probability. For any $z \in X$ we cannot have any points in $D_z(\rho)$ that are not in X , by the minimality of $\|ab\|$, and so in particular C is empty, i.e. Condition 2 is met. Finally, since $ab \notin G$ and since $D_a(\rho) \cap D_b(\rho)$ is empty by the minimality of $\|ab\|$, there must be at least k points in at least one of A or B , so Condition 3 is met. \square

We will show that if $k = c \log n$ and $c > 1.0293$, then with high probability no quadruple forms a component set-up, at which point Lemma 24 tells us there will be no small component in G with high probability.

Lemma 25. *If:*

$$c > \log \left(\frac{8\pi + 3\sqrt{3}}{2\pi + 3\sqrt{3}} \right)^{-1} \approx 1.0293$$

and $k = c \log n$, then, with high probability, no quadruple (a, b, x_l, x_r) with all of a, b, x_l and x_r at least $d\sqrt{\log n}$ from the boundary of S_n form a component set-up.

Proof. We will show that if we pick four points in S_n ; a, b, x_l and x_r that are all within $d\sqrt{\log n}$ of a (i.e. meet Condition 1 of being a component set-up), then the probability, $p(n)$, that they meet Conditions 2 and 3 of being a component set-up decays as at least $n^{-(1+\varepsilon)}$ for some $\varepsilon > 0$. Then, since there are only $O(n)$ points in S_n in total (with high probability), and since all four points are within $d\sqrt{\log n}$ of a , Lemma 3 tells us that there are only $O(n(\log n)^3)$ choices for such a system, and so, with high probability, no four points form a component set-up.

Since x_l and x_r are at least $d\sqrt{\log n}$ from the boundary of S_n , and $\rho = \|ab\| \leq d\sqrt{\log n}$, we have that $|C| = \pi\rho^2$. We also know that $|A|, |B| \leq (\pi/3 + \sqrt{3}/2)\rho^2$, and so, by Lemma 9:

$$\begin{aligned}
p(n) &\leq \mathbb{P}(\#C = 0 \text{ and } \#A \geq k) + \mathbb{P}(\#C = 0 \text{ and } \#B \geq k) \\
&\leq \left(\frac{|A|}{|A \cup C|}\right)^k + \left(\frac{|B|}{|B \cup C|}\right)^k \\
&\leq 2 \left(\frac{(\pi/3 + \sqrt{3}/2)\rho^2}{\pi\rho^2 + (\pi/3 + \sqrt{3}/2)\rho^2}\right)^k \\
&= 2 \left(\frac{2\pi + 3\sqrt{3}}{8\pi + 3\sqrt{3}}\right)^k \\
&= 2 \exp\left(-c \log\left(\frac{8\pi + 3\sqrt{3}}{2\pi + 3\sqrt{3}}\right) \log n\right) \tag{27}
\end{aligned}$$

If $c > \log\left(\frac{8\pi + 3\sqrt{3}}{2\pi + 3\sqrt{3}}\right)^{-1}$, then (27) is at most $2n^{-(1+\varepsilon(c))}$ for some $\varepsilon(c) > 0$, and so we are done. \square

We now rule out having a component set-up near the edge of S_n , and so having a small component near the edge of S_n . The bound we prove here will also be strong enough to rule out the edge case in our stronger bound on the connectivity threshold that we give in the next section.

Lemma 26.

1. If $c > 0$ and $k = c \log n$, then with high probability there is no component set-up containing a point within $2d\sqrt{\log n}$ of a corner of S_n .
2. If $c > 0.8343$ and $k = c \log n$, then with high probability there is no component set-up containing a point within $d\sqrt{\log n}$ of any edge of S_n .

Proof. The proof proceeds almost exactly as in the previous lemma. We again pick our four points a, b, x_l and x_r with b, x_l and x_r within $d\sqrt{\log n}$ of a and bound the probability that they meet Conditions 2 and 3 of forming a component-set-up. We write $p_c(n)$ and $p_e(n)$ for the probabilities of these events for a quadruple near a corner and an edge respectively.

Part 1 The number of such quadruples with at least one point within $2d\sqrt{\log n}$ of a corner is $O((\log n)^4)$. We show that $p_c(n)$ decays as at least $n^{-\varepsilon}$, for some $\varepsilon > 0$.

We will have that $|A|, |B| \leq (\pi/3 + \sqrt{3}/2)\rho^2$ (where again $\rho = \|ab\|$).

If one of our points is within $d\sqrt{\log n}$ of a corner of S_n we must still have

$|C| \geq \pi/4$, and so, using Lemma 9:

$$\begin{aligned}
p_c(n) &\leq \mathbb{P}(\#C = 0 \text{ and } \#A \geq k) + \mathbb{P}(\#C = 0 \text{ and } \#B \geq k) \\
&\leq \left(\frac{|A|}{|A| + |C|} \right)^k + \left(\frac{|B|}{|B| + |C|} \right)^k \\
&< 2 \left(\frac{(\pi/3 + \sqrt{3}/2)\rho^2}{(\pi/4)\rho^2 + (\pi/3 + \sqrt{3}/2)\rho^2} \right)^{c \log n} \\
&< 2n^{-0.3439c}
\end{aligned} \tag{28}$$

And thus for any $c > 0$ the exponent of (28) is strictly less than zero, and so with high probability there are no small components containing a point within $d\sqrt{\log n}$ of any corner of S_n .

Part 2 The number of such quadruples with at least one point within $d\sqrt{\log n}$ of an edge is $O(\sqrt{n}(\log n)^3)$. We show that $p_e(n)$ decays as at least $n^{-(1/2+\varepsilon)}$, for some $\varepsilon > 0$. If none of our points are within $2d\sqrt{\log n}$ of a corner, but at least one is within $2d\sqrt{\log n}$ of an edge, then $|C| \geq \frac{\pi}{2}\rho^2$ (either we have all of one of the half disks $D_{x_l}^l$ and $D_{x_r}^r$ or at least half of each), and so:

$$\begin{aligned}
p_e(n) &\leq \left(\frac{|A|}{|A| + |C|} \right)^k + \left(\frac{|B|}{|B| + |C|} \right)^k \\
&< 2 \left(\frac{(\pi/3 + \sqrt{3}/2)\rho^2}{(\pi/2)\rho^2 + (\pi/3 + \sqrt{3}/2)\rho^2} \right)^{c \log n} \\
&< 2n^{-0.5993c}
\end{aligned} \tag{29}$$

For any $c > 0.8343$ the exponent of (29) is strictly less than $-\frac{1}{2}$ and so we are done. □

Putting together Lemmas 25 and 26, and applying Lemma 24 and Proposition 23, we have:

Proposition 27. *Let $p(n)$ be the probability that $G_{n,k}$ is disconnected, then, provided $k = c \log n$ and:*

$$c > \log \left(\frac{8\pi + 3\sqrt{3}}{2\pi + 3\sqrt{3}} \right)^{-1} \approx 1.0293$$

we have:

$$p(n) \rightarrow 0, \text{ as } n \rightarrow \infty$$

4.2 The Size of Small Components and an Improved Bound

The previous section gives a reasonably good upper bound on the connectivity threshold for $G_{n,k}$, so that we know if $k > 1.0293 \log n$, then $G_{n,k}$ is connected with high probability. The best lower bound known is that if $k < 0.7209 \log n$

then $G_{n,k}$ is disconnected with high probability, which follows from Balister, Bollobás, Sarkar and Walter's bound on the directed model [1]. This leaves the question: could the connectivity threshold be exactly $k = \log n$? We show that this hypothesis, which was conjectured originally by Xue and Kumar for the original undirected model [7], and is true in the Gilbert model, does not hold here, thus further disproving their conjecture, since the threshold for the strict undirected model must be at least as high as that in the original undirected model. In particular we show that if $k > 0.9684 \log n$ then G is connected with high probability.

To show this improved bound, we first show that the small components in G (i.e. of diameter $\Phi(\log n)$) contain far fewer than k points as k approaches the lower bound on the connectivity threshold, and then use this to improve our upper bound. One major tool that we use in this section is an isoperimetric argument. As in [6] this will allow us to bound the empty area around any small component as a function of how much space that component takes up. We use the isoperimetric theorem in its following form, which is a consequence of the Brunn-Minkowski inequality, see e.g. [3]. Part 2 of the Lemma follows from an easy reflection argument.

Lemma 28.

1. *For any $\lambda > 0$ the subset A of the plane of area λ that minimises the area of the δ -blowup, $A(\delta)$ (the subset of the plane within δ of any location in A), is the disc of area λ .*
2. *The subset A on the half plane E^+ of area λ that minimises the area of the intersection of $A(\delta)$ and E^+ is the half disc of area λ centred along the edge of E^+ .*

To use Lemma 28, we follow [6] and tile S_n with a fine square grid. We can then look at the number of tiles that a small component hits to give a bound on the empty area around it. To be precise:

We set $M = 20000d$ (a large enough value to gain a good result) and tile S_n with small squares of side length $s = \sqrt{\log n}/M$. We form a graph \widehat{G} on these tiles by joining two tiles whenever the distance between their centres is at most $2d\sqrt{\log n}$. We call a pointset *bad* if any of the following hold (and *good* otherwise):

1. there exist two points that are joined in G but the tiles containing these points are not joined in \widehat{G} ,
2. there exist two points at most distance $\frac{1}{d}\sqrt{\log n}$ apart that are not joined,
3. there exists a half-disc based at a point of G of radius $d\sqrt{\log n}$ that is contained entirely within S_n and contains no (other) point of G ,
4. there exists two components in $G_{n,k}$ with Euclidean diameter at least $d\sqrt{\log n}$,
5. there exists a component of diameter at most $d\sqrt{\log n}$ containing a vertex within distance $2d\sqrt{\log n}$ of a corner of S_n .
6. there exists two different components X and Y such that an edge in component X crosses an edge in component Y .

Note that unlike in [6], we do not insist that a small component cannot be near an edge of S_n , but only that it can't be near a corner, since our Lemma 26 is not strong enough to rule out the existence of small components near the edge of S_n around the lower bound on the connectivity threshold ($k = 0.7209 \log n$).

Lemma 29. *If $k = c \log n$ and $c > 0.7102$, then with high probability the configuration is good.*

Proof.

- By our choice of d and Lemma 3 Conditions 1, 2 and 3 hold with high probability.
- For $k > 0.7102 \log n$, Proposition 23 ensures Condition 4 holds with high probability.
- Lemma 26 part 1 ensures Condition 5 holds with high probability.
- For $k > 0.7102 \log n$, Theorem 1 ensures Condition 6 holds with high probability.

Since each condition holds with high probability, they will all hold together with high probability, and so the configuration will be good with high probability. \square

We will consider what can happen around a small component once we know which tiles the component meets. We make the following definitions:

Definition 8. *Given two points, a, b , and a collection of tiles Y with $a \in Y$ and $b \notin Y$, we define, as before, $\rho = \|ab\|$ and $A = (D_a(\rho) \setminus D_b(\rho)) \cap S_n$, and define the regions:*

- Z to be all tiles not in Y with their centre within $\rho - \sqrt{2}s$ of the centre of a tile in Y ,
- B' to be $D_b(\rho) \setminus (D_a(\rho) \cup Y \cup Z)$, and
- Y' to be the tiles in Y that have their centre within $\rho + \sqrt{2}s$ of a (so that the tiles in Y that meet the region A defined previously are all in Y').

See Figure 10 for an illustration.

We can use these new regions to form an analogous version of Lemma 24.

Lemma 30. *If G contains a component, X , of diameter at most $d\sqrt{\log n}$, then with high probability there will be some triple (a, b, Y) such that:*

1. *The diameter of Y is at most $d\sqrt{\log n} + 2\sqrt{2}s$,*
2. *b is within $d\sqrt{\log n}$ of a ,*
3. *$\#Z = 0$, and*
4. *at least one of $\#Y'$ and $\#B'$ is at least k .*

Proof. Given a component X , we set Y to be the set of tiles that contain a point in X , and a and b to be the pair of points such that $a \in X$, $b \notin X$ that minimise $\rho = \|ab\|$.

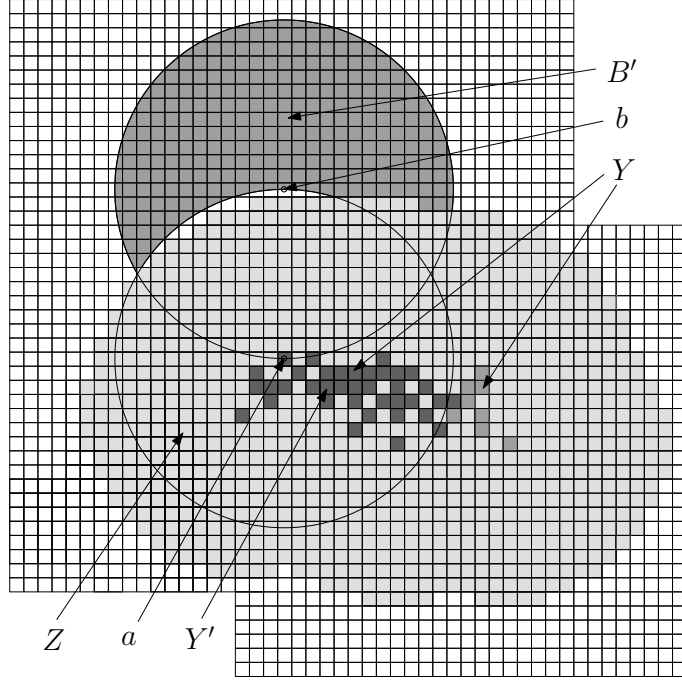


Figure 10: The points a and b , and the regions Y , Y' , Z and B' .

- Condition 1 holds as $\text{diam}(Y) \leq \text{diam}(X) + 2\sqrt{s}$.
- Condition 2 follows from Lemma 3.
- Condition 3 follows since no point outside of X can be within ρ of a point in X and every tile of Y contains a point in X .
- Condition 4 follows since ab is not an edge of G , and every location in any tile with its centre within $\rho - \sqrt{2}$ of the centre of a tile containing a point $x \in X$ must be within ρ of x .

□

The Isoperimetric Theorem (Lemma 28) allows us to bound the area of Z in terms of the area of Y :

Lemma 31. *For a triple (a, b, Y) , if no tile of Y is within $d\sqrt{\log n}$ of the edge of S_n then, writing $r = \rho - \sqrt{2}s > (1 - 10^{-4})\rho$ (where again $\rho = \|ab\|$), we have:*

$$|Z| \geq \pi r^2 + 2r\sqrt{\pi|Y|}$$

If Y does contain a tile within $d\sqrt{\log n}$ of the edge of S_n , but no tile within $2d\sqrt{\log n}$ of a corner then:

$$|Z| \geq \frac{\pi}{2}r^2 + r\sqrt{\pi|Y|}$$

Proof. The Isoperimetric Theorem tells us that the area of $|Z|$ is at least what it would be if Y was a disk and Z was its r blow-up. In this case:

$$\text{radius}(Y) = \sqrt{|Y|/\pi}$$

and so:

$$\begin{aligned} |Z| &\geq \pi \left(r + \sqrt{\pi/|Y|} \right)^2 - |Y| \\ &= \pi r^2 + 2r\sqrt{\pi|Y|} \end{aligned}$$

The second part follows in exactly the same way, using part 2 of our version of the Isoperimetric Theorem. \square

With this machinery in place, we can now proceed to prove that as k nears the connectivity threshold, all small components are very small, i.e. of size much less than k . The proof works in two parts: We first prove that, with high probability, no triple (a, b, Y) has $\#Y' \geq k$ and $\#Z = 0$ for $k \geq 0.7209 \log n$. This allows us to conclude that if G contains a small component, then with high probability some triple (a, b, Y) has $B' \geq k$ and $\#Z = 0$ by Lemma 30. We then use this to bound the size of any small component by showing that no triple (a, b, Y) has $\#B' \geq k$, $\#Z = 0$ and $\#Y \geq 0.309k$ with high probability.

Lemma 32. *If $c > 0.7209$ and $k = c \log n$, then with high probability, no triple (a, b, Y) meeting Condition 1-4 of Lemma 30 has $\#Y' \geq k$.*

Proof. Let $p_A(n)$ be the probability that a given triple (a, b, Y) with no part of Y within $d\sqrt{\log n}$ of the boundary of S_n and meeting Conditions 1 and 2 of Lemma 30 also meets Conditions 3 and has $\#Y' \geq k$. Let $p_{A'}(n)$ be this same probability when Y does contain a tile within $d\sqrt{\log n}$ of the boundary of S_n .

Case 1 Y does not contain a tile within $d\sqrt{\log n}$ of the boundary of S_n :

There will be $O(n)$ choices for the point a , and once a has been chosen, there are only $O(\log n)$ choices for b (since it is within $d\sqrt{\log n}$ of a), and only a (large) constant number of choices for Y , since Y can only include tiles from the fixed collection of $16(dM)^2$ tiles nearest to a (i.e. the tiles within $d\sqrt{\log n}$ of a). Thus there are $O(n \log n)$ possible triples (a, b, Y) meeting Conditions 1 and 2 of Lemma 30.

We show that $p_A(n)$ decays at least as fast as $n^{-(1+\varepsilon)}$.

By Lemma 31:

$$\begin{aligned} |Z| &\geq \pi r^2 + 2r\sqrt{\pi|Y|} \\ &\geq \pi r^2 + 2r\sqrt{\pi|Y'|} \end{aligned}$$

where $r = \rho - \sqrt{2}s > (1 - 10^{-4})\rho$.

Since every tile of Y' contains a location within $\rho + 2\sqrt{2}s$ of a , and no tile in Y' contains a location within $\rho - 2\sqrt{2}s$ of b , we have:

$$\begin{aligned} |Y'| &\leq \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \right) \rho^2 + \pi \left((\rho + 2\sqrt{2}s)^2 - \rho^2 \right) \\ &< \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} + \frac{\pi}{1000} \right) \rho^2 \end{aligned} \tag{30}$$

If (a, b, Y) meets Condition 3 of Lemma 30 (i.e. has $\#Z = 0$), and $\#Y' \geq k$, then by Lemma 9:

$$\begin{aligned} p_A(n) &\leq \left(\frac{|Y'|}{|Y'| + |Z|} \right)^k \\ &\leq \left(\frac{|Y'|}{\pi r^2 + 2r\sqrt{\pi|Y'|} + |Y'|} \right)^k \\ &= \exp \left(-c \log \left(\frac{\pi r^2 + 2r\sqrt{\pi|Y'|} + |Y'|}{|Y'|} \right) \log n \right) \end{aligned} \quad (31)$$

Maximising (31) over the range $0 < |Y'| < \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} + \frac{\pi}{1000} \right) \rho^2$, we achieve a maximum of $n^{-1.18\dots}$ (when $|Y'|$ is maximal). Thus, with high probability, we will have no system with $\#Y' \geq k$.

Case 2 Y does contain a tile within $d\sqrt{\log n}$ of the boundary of S_n :

We will have $O(n^{1/2})$ choices for a , and the same argument as in the previous case shows that there are $O(n^{1/2} \log n)$ such triples meeting Conditions 1 and 2 of Lemma 30 that also have some tile of Y within $d\sqrt{\log n}$ of the boundary of S_n .

We show that $p_{A'}(n)$ decays as at least $n^{-(1/2+\epsilon)}$.

Here Lemma 31 only ensures $|Z| \geq \frac{1}{2}\pi r^2 + r\sqrt{\pi|Y'|}$. Equation (30) still holds and (31) becomes:

$$p'_{A'}(n) \leq \exp \left(-c \log \left(\frac{\frac{1}{2}\pi r^2 + r\sqrt{\pi|Y'|} + |Y'|}{|Y'|} \right) \log n \right) \quad (32)$$

Maximising (32) over the range $0 < |Y'| < \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} + \frac{\pi}{1000} \right) \rho^2$, we achieve a maximum of $n^{-0.81\dots}$ (again when $|Y'|$ is maximal). Thus again, with high probability, we will have no system with $\#Y' \geq k$, and thus with high probability no small component has $\#Y' \geq k$.

□

Lemma 32 tells us that, with high probability, as k approaches the connectivity threshold, every triple (a, b, Y) that corresponds exactly to a small component, will have $\#B' \geq k$ (i.e. we can change Condition 4 in Lemma 30 (from $\#A \geq k$ or $\#B' \geq k$) to simply $\#B' \geq k$ (denote this Condition 4'), and the Lemma will stay true). We use this to strengthen the previous argument and show that in fact there are far fewer than k points in the whole of any small component, but first need a result about how dense two disjoint regions can be simultaneously. The following is a result about the Poisson process that is a slight alteration of Lemma 6 from [6] which goes through by exactly the same proof:

Lemma 33. *If X, Y and Z are three regions with $|X| \leq |Y \cup Z|$, $|Y| \leq |X \cup Z|$ and $X \cap Y = \emptyset$, then, writing E for the event that $\#X \geq mk$, $\#Y \geq k$ and $\#Z = 0$, we have:*

$$\mathbb{P}(E) \leq \left(\frac{2|X|}{|X| + |Y| + |Z|} \right)^{mk} \left(\frac{2|Y|}{|X| + |Y| + |Z|} \right)^k \quad (33)$$

We can now show, by a similar argument to Lemma 32:

Proposition 34. *Let $c > 0.7209$ and $k = c \log n$. Then with high probability no small component contains more than $0.309k$ points of G .*

Proof. If G contains a small component with at least $0.309k$ points, then with high probability there will be some triple (a, b, Y) that meets Conditions 1–3 of Lemma 30, Condition 4' and $\#Y \geq 0.309k$. We write p_X for the probability that a triple (a, b, Y) meeting Conditions 1 and 2 meets the rest of these conditions when Y contains no tile within $d\sqrt{\log n}$ of the boundary of S_n and $p_{X'}$ for the same probability when Y does contain such a tile. As in Lemma 32 it suffices to show that p_X decays at least as fast as $n^{-1-\varepsilon}$ and $p_{X'}$ decays as at least $n^{-1/2-\varepsilon}$ for some $\varepsilon > 0$ to complete the proof.

We wish to apply Lemma 33, but need to check the conditions of the Lemma first:

1. The condition $|B'| \leq |Y \cup Z|$ follows as $|Z| \geq \pi r^2 \approx 3.14\rho^2$ and $|B'| \leq (\pi/3 + \sqrt{3}/2)\rho^2 \approx 1.91\rho^2$, and so $|Z| \geq |B'|$.
2. The condition that $B' \cap Y = \emptyset$ follows by definition.
3. The condition $|Y| < |B' \cup Z|$: By Lemma 31, $|Z| \geq \pi r^2 + 2r\sqrt{\pi|Y|}$ when Y contains no tile within $d\sqrt{\log n}$ of the edge of S_n and $|Z| \geq \pi r^2/2 + r\sqrt{\pi|Y|}$ when Y does. Solving $|Y| > \pi r^2 + 2r\sqrt{\pi|Y|}$ and $|Y| > \pi r^2/2 + r\sqrt{\pi|Y|}$, we gain that $|Y| > 11.72\rho^2$ and $|Y| > 5.861\rho^2$ respectively. Thus, so long as $|Y| \leq 11.7\rho^2$ in the centre case, and $|Y| \leq 5.86\rho^2$ in the edge case, $|Y| < |Z|$, and so the condition holds. When Y exceeds these bounds, we cannot apply Lemma 33, but instead note that, for Y in this range:

$$\begin{aligned}
p_X &\leq \mathbb{P}(\#Z = 0 \text{ and } \#B' \geq k) \\
&\leq \left(\frac{|B'|}{|B'| + |Z|} \right)^k \\
&\leq \left(\frac{(\pi/3 + \sqrt{3}/2)\rho^2}{(\pi/3 + \sqrt{3}/2)\rho^2 + \pi r^2 + 2r\sqrt{\pi|Y|}} \right)^k \\
&< \left(\frac{\pi/3 + \sqrt{3}/2}{4\pi/3 + \sqrt{3}/2 + 2\sqrt{11.7}} \right)^k \\
&< n^{-1.58}
\end{aligned} \tag{34}$$

By an exact analogy in the edge case, when $|Y| > 5.86\rho^2$, we find that:

$$p_{X'} < n^{-1.01} \tag{35}$$

Thus, for $c \geq 0.7209$, and recalling that $r > (1 - 10^{-4})\rho$:

$$\begin{aligned}
p_X &\leq \mathbb{P}(|Y| \leq 11.7\rho^2) \mathbb{P}(\#Z = 0, \#B' \geq k, \#Y \geq 0.309k \mid |Y| \leq 11.7\rho^2) \\
&\quad + \mathbb{P}(|Y| > 11.7\rho^2) n^{-1.58} \\
&\leq \max_{|Y| \leq 11.7\rho^2} \left(\frac{2|Y|}{|B'| + |Y| + |Z|} \right)^{0.309k} \left(\frac{2|B'|}{|B'| + |Y| + |Z|} \right)^k + n^{-1.58} \\
&\leq \max_{|Y| \leq 11.7\rho^2} \frac{(2|Y|)^{0.309k} (2(\pi/3 + \sqrt{3}/2)\rho^2)^k}{\left((\pi/3 + \sqrt{3}/2)\rho^2 + |Y| + \pi r^2 + 2r\sqrt{\pi|Y|} \right)^{1.309k}} + n^{-1.58} \\
&\leq \max_{|Y| \leq 11.7\rho^2} \frac{(2|Y|)^{0.309k} (2(\pi/3 + \sqrt{3}/2)\rho^2)^k}{\left((\pi/3 + \sqrt{3}/2)\rho^2 + |Y| + \pi r^2 + 2r\sqrt{\pi|Y|} \right)^{1.309k}} + n^{-1.58} \quad (36)
\end{aligned}$$

Maximising the first term over the range $0 \leq |Y| \leq 11.7\rho^2$, we find that the first term of (36) achieves a maximum of $n^{-1.0001\dots}$ when $|Y| = 0.6069\rho^2 \dots$

Similarly we have:

$$\begin{aligned}
p_{X'} &\leq \mathbb{P}(|Y| \leq 5.86\rho^2) \mathbb{P}(\#Z = 0, \#B' \geq k, \#Y \geq 0.309k \mid |Y| \leq 5.86\rho^2) \\
&\quad + \mathbb{P}(|Y| > 5.86\rho^2) n^{-1.01} \\
&\leq \max_{|Y| \leq 5.86\rho^2} \frac{(2|Y|)^{0.309k} (2(\pi/3 + \sqrt{3}/2)\rho^2)^k}{\left((\pi/3 + \sqrt{3}/2)\rho^2 + |Y| + \pi r^2/2 + r\sqrt{\pi|Y|} \right)^{1.309k}} + n^{-1.01} \quad (37)
\end{aligned}$$

Maximising the first term over the range $0 \leq |Y| \leq 5.86\rho^2$, we find that the first term of (37) achieves a maximum of $n^{-0.593\dots}$ when $|Y| = 0.601\rho^2$.

Thus, with high probability, no triple (a, b, Y) has $\#Y \geq 0.309k$, $\#B' \geq k$ and $\#Z = 0$, and so with high probability there is no small component containing more than $0.309k$ points. \square

We will use this result to prove a stronger bound on the connectivity threshold. The idea is to show that, with high probability, any triple (a, b, Y) which meets Conditions 1-3 of Lemma 30, Condition 4' and has $\#Y \leq 0.309k$, which we know happens with high probability if G contains a small component, will have another point, β , in neither B' nor Y , but is within 1.0767ρ of a such that $\overrightarrow{a\beta}$ is an out edge, but $\overrightarrow{\beta a}$ is not. There must then be a dense region around β , and we can use this to improve our bound on the connectivity threshold. More precisely we will show that there are k points in the following region:

Definition 9. *Given the system (a, b, β, Y) with a, b and Y as usual and $\beta \notin Y \cup B'$, we define the region (shown in Figure 11):*

$$B^* = \left[(D_\beta(\|a\beta\|) \cap B') \cup (D_\beta(\|a\beta\|) \setminus D_a(\|a\beta\|)) \right] \setminus (Y \cup Z)$$

We introduce one more piece of notation, and then prove that there will be a suitable β with high probability.

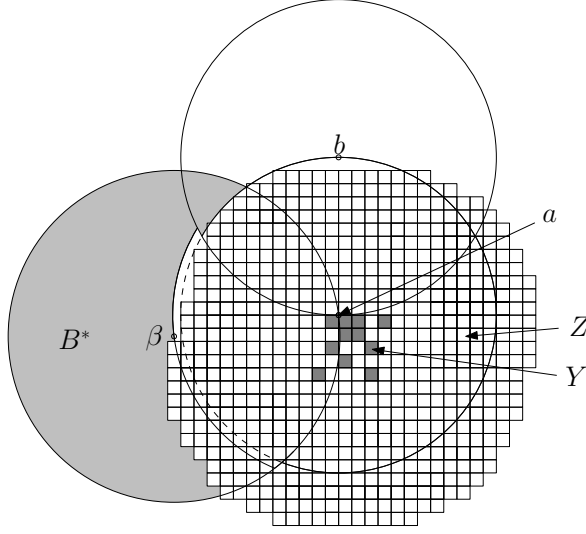


Figure 11: The point β and the region B^* .

Definition 10. Given $\lambda > \rho$, we write $B(\lambda) = B' \cap D_a(\lambda)$ and $A(\lambda) = D_a(\lambda) \setminus (D_a(\rho) \cup B)$. See Figure 12.

The following lemma tells us that with high probability, if G contains a small component, then we can find a suitable point β .

Lemma 35. If $k > 0.9684 \log n$ and G contains a component of diameter at most $d\sqrt{\log n}$, then with high probability there is some quadruple (a, b, β, Y) such that:

1. The diameter of Y is at most $d\sqrt{\log n} + 2\sqrt{2}s$,
2. b is within $d\sqrt{\log n}$ of a ,
3. $\#Z = 0$,
4. $\#B' \geq k$,
5. $\#Y \leq 0.309k$,
6. Y contains no tile within $d\sqrt{\log n}$ of the boundary of S_n ,
7. $\beta \in A(1.0767\rho)$ and
8. $\#B^* \geq k$.

Proof. Given a small component, X , we take Y to be exactly the tiles that meet X and a and b to be the pair such that $a \in X$, $b \notin X$ and $\|ab\|$ is minimal, all as usual. Then Conditions 1–3 are met with high probability by Lemma 30, Condition 4 is met by Lemma 32, Condition 5 is met by Proposition 34 and Condition 6 is met by Lemma 26. We take β to be the point outside of $B' \cup Y \cup Z$ that is closest to a .

To show Condition 7 holds with high probability we show that no triple (a, b, Y) meeting Conditions 1 and 2 has both:

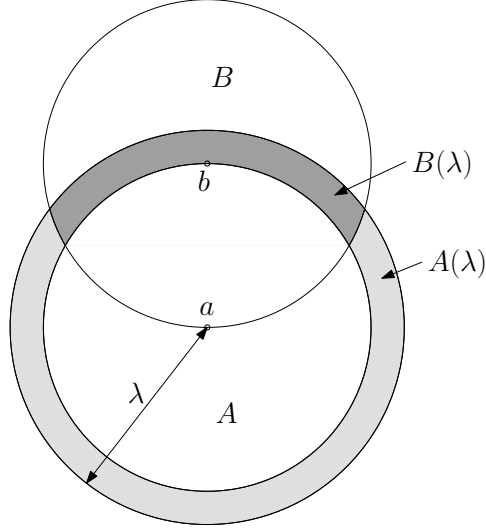


Figure 12: The region $A(\lambda)$ and $B(\lambda)$.

1. $\#B' \geq k$ and,
2. $\#(Z \cup A(1.0767\rho) \setminus Y) = 0$.

If, with high probability, this does not occur, then with high probability there will be some point in $A(1.0767\rho)$, and so in particular $\beta \in A(1.0767\rho)$.

We write E_1 for the event that a particular triple has $\#B' \geq k$, $\#(Z \cup A(1.0767\rho) \setminus Y) = 0$ and meets Conditions 1 and 2. We know that $|B'| \leq \pi/3 + \sqrt{3}/2$ and, by Lemma 9:

$$\mathbb{P}(E_1) \leq \left(\frac{|B'|}{|B'| + |Z \cup A(1.0767\rho) \setminus Y|} \right)^k$$

Thus $\mathbb{P}(E_1)$ will be maximised when B' is maximised and $|(A(1.0767\rho) \cup Z) \setminus Y|$ is minimised. By the Isoperimetric Theorem, this will occur when Y is the small disk centred on a whose r blow-up just covers $A(1.0767\rho)$. In this case:

$$\text{radius}(Y) = 1.0767\rho - r \leq 0.0768\rho$$

And so, omitting the trivial but tedious calculations to evaluate $|A(1.0767\rho)|$:

$$\begin{aligned} |(Z \cup A(1.0767\rho) \setminus Y)| &\geq |D_a(\rho)| + |A(1.0767\rho)| - \pi(0.0768\rho)^2 \\ &> 3.4602\rho^2 \end{aligned} \tag{38}$$

Thus:

$$\begin{aligned}
\mathbb{P}(E_1) &\leq \left(\frac{|B'|}{|B'| + |Z \cup A(1.0767\rho) \setminus Y|} \right)^k \\
&\leq \left(\frac{(\pi/3 + \sqrt{3}/2)\rho^2}{(\pi/3 + \sqrt{3}/2)\rho^2 + 3.4602\rho^2} \right)^{0.9684 \log n} \\
&< n^{-1.00004}
\end{aligned} \tag{39}$$

Since there are only $O(n \log n)$ such systems, (39) tells us that E_1 will not occur for any of them with high probability, and so Condition 7 holds with high probability.

To show Condition 8 holds with high probability we first show that $\overrightarrow{a\beta}$ is an out edge with high probability. Then since $a\beta$ cannot be an edge, $D_\beta(\|a\beta\|)$ must contain k points, and we finish the proof by showing that the nearest k of these to β will all lie in B^* with high probability.

If some small component did not have $\overrightarrow{a\beta}$ being an out-edge, then, since $\#Y \leq 0.309k$, there would be at least $(1 - 0.309)k = 0.691k$ points in $B(\|a\beta\|) \subset B(1.0767\rho)$. Then there would be some triple (a, b, Y) with $\#B(1.0767\rho) \geq 0.691k$ and $\#Z = 0$. We write E_2 for the event that a given triple meeting Conditions 1 and 2 has $\#B(1.0767\rho) \geq 0.691k$ and $\#Z = 0$. Calculations show that $|B(1.0767\rho)| \leq 0.1632\rho^2$, and we know that $|Z| \geq \pi r^2$, thus:

$$\begin{aligned}
\mathbb{P}(E_2) &\leq \left(\frac{|B(1.0767\rho)|}{|B(1.0767\rho) \cup Z|} \right)^{0.691k} \\
&\leq \left(\frac{0.1632\rho^2}{0.1632\rho^2 + \pi r^2} \right) \\
&< n^{-2.3}
\end{aligned} \tag{40}$$

Thus, E_2 does not occur for any triple (a, b, Y) with high probability, and so $\overrightarrow{a\beta}$ will be an out edge with high probability.

This tells us that $D_\beta(\|a\beta\|)$ must contain k points, and we know that none of these points are in $Z \cup A(\|a\beta\|)$. Thus they must lie in $B^* \cup Y$. We complete the proof by showing that with high probability none of the k -nearest neighbours of β lie in Y .

If there were a point, γ , in $D_\beta(\|a\beta\|) \cap Y$ such that γ was one of the k -nearest neighbours of β , then there must be k points within $D_\gamma(\|\beta\gamma\|)$ since $\beta\gamma$ is not an edge of G . At most $0.309k$ of these can be in Y by Proposition 34, and no other points can be within $D_\gamma(\rho)$. Thus there must be at least $0.691k$ points within $D_\gamma(\|\beta\gamma\|) \setminus (D_\gamma(\rho) \cup Y \cup Z) \subset D_\gamma(\|\beta\gamma\|) \setminus (D_\gamma(\rho) \cup Z)$.

Given a system (a, b, β, γ, Y) with a, b , and Y as before, $\beta \in A(1.0767\rho)$ and $\gamma \in D_\beta(\|a\beta\|) \cap Y$, we write E_3 for the event that $\#Z = 0$ and $\#D_\gamma(\|\beta\gamma\|) \setminus (D_\gamma(\rho) \cup Z) \geq 0.691k$. We know $|Z| \geq \pi r^2$ and $|D_\gamma(\|\beta\gamma\|) \setminus (D_\gamma(\rho) \cup Z)| \leq$

$\pi(1.0767^2 - 1)\rho^2$, thus:

$$\begin{aligned}\mathbb{P}(E_3) &\leq \left(\frac{|D_\gamma(\|\beta\gamma\|) \setminus (D_\gamma(\rho) \cup Z)|}{|Z \cup D_\gamma(\|\beta\gamma\|) \setminus (D_\gamma(\rho) \cup Z)|} \right)^{0.691k} \\ &\leq \left(\frac{\pi(1.0767^2 - 1)\rho^2}{\pi(r^2 + 1.0767^2\rho^2 - \rho^2)} \right)^{0.691k} \\ &< n^{-1.3}\end{aligned}\tag{41}$$

Thus, with high probability, E_3 does not occur for any such system (a, b, β, γ, Y) , and so in particular none of the k nearest neighbours of β will be in Y with high probability, and so we will have $\#B^* \geq k$ with high probability as required. \square

We can now prove our stronger bound on the connectivity threshold, but first state a result about the probability of two intersecting regions being dense, which can be read out of the proof of Theorem 15 of [1].

Lemma 36. *Let A_1, A_2, A_3 and A_4 be four disjoint regions of S_n and let $n_i = \#A_i$. Then, so long as $|A_1| \leq |A_3| < 2|A_1|$, we have:*

$$\mathbb{P}(n_1 + n_2 \geq k, n_2 + n_3 \geq k \text{ and } n_4 = 0) \leq \mu^{-k} n^{o(1)}$$

where μ is the solution to:

$$\sum_{i=1}^4 |A_i| = \mu |A_2| + \sqrt{4\mu |A_1| |A_3|}$$

\square

Theorem 2 *If $k = c \log n$ and $c > 0.9684$, then G is connected with high probability.*

Proof. We know that if G contains a small component then with high probability there will be a system (a, b, β, Y) meeting all the conditions of Lemma 35. We show that for $c > 0.9684$ no such system meets all these conditions with high probability.

Given a system (a, b, β, Y) meeting Conditions 1, 2, 6 and 7 of Lemma 35 (so that there are $O(n(\log n)^2)$ such systems), we write E for the event $\#B' \geq k$ and $\#B^* \geq k$ and set:

$$\begin{aligned}B_1 &= B' \setminus B^* \\ B_2 &= B' \cap B^* \\ B_3 &= B^* \setminus B'\end{aligned}$$

We write $n_i = \#B_i$ for $(i = 1, 2, 3)$, $n_4 = \#Z$, then E is the event $n_1 + n_2 \geq k$, $n_2 + n_3 \geq k$ and $n_4 = 0$.

We wish to apply Lemma 36, but need to make sure that either $|B_1| \leq |B_3| < 2|B_1|$ or $|B_3| \leq |B_1| < 2|B_3|$. We know that $|B'| \leq (\frac{\pi}{3} + \frac{\sqrt{3}}{2})\rho^2$ and calculations show that $|B^*| < 2.31\rho^2$ and $|B' \cap B^*| < 0.6515\rho^2$. From this it is easily checked that the conditions will hold unless at least one of $|B^*|$ or $|B'|$ is small whilst the other is large, in particular, at least one of $|B_1| \leq |B_3| < 2|B_1|$

or $|B_3| \leq |B_1| < 2|B_3|$ will hold so long as $|B^*| \geq 1.73\rho^2$ and $|B'| \geq 1.73\rho^2$. When one of these does not hold, we note that:

$$\mathbb{P}(E) \leq \mathbb{P}(\#Z = 0 \text{ and } \#B' = 0)$$

And:

$$\mathbb{P}(E) \leq \mathbb{P}(\#Z = 0 \text{ and } \#B^* = 0)$$

And apply Lemma 9. Thus we have:

$$\begin{aligned} \mathbb{P}(E) &\leq \mathbb{P}(|B'|, |B^*| \geq 1.73\rho^2) \mathbb{P}(E | |B'|, |B^*| \geq 1.73\rho^2) \\ &\quad + \mathbb{P}(|B'| < 1.73) \mathbb{P}(E | |B'| < 1.73) \\ &\quad + \mathbb{P}(|B^*| < 1.73) \mathbb{P}(E | |B^*| < 1.73) \\ &\leq \max_{|B'|, |B^*| \geq 1.73\rho^2} \mu^{-k} n^{o(1)} + \max_{|B'| < 1.73\rho^2} \left(\frac{|B'|}{|B'| + |Z|} \right)^k \\ &\quad + \max_{|B^*| < 1.73\rho^2} \left(\frac{|B^*|}{|B^*| + |Z|} \right)^k \\ &< \max_{|B'|, |B^*| \geq 1.73\rho^2} \mu^{-k} n^{o(1)} + 2 \left(\frac{1.73\rho^2}{1.73\rho^2 + \pi r^2} \right)^k \\ &\leq \max_{|B'|, |B^*| \geq 1.73\rho^2} \mu^{-k} n^{o(1)} + 2n^{-1.01} \end{aligned} \tag{42}$$

where:

$$|Z| + \sum_i |B_i| = \mu |B_2| + \sqrt{4\mu |B_1| |B_3|} \tag{43}$$

Thus $\mathbb{P}(E)$ will be maximised exactly when μ is minimised, which will be when B^* overlaps with B' as much as possible and $|B'|$ and $|B^*|$ are maximal. This will happen when β is located at $\partial D_a(1.0767\rho) \cap \partial B'$. Calculating μ in this case yields $\mu > 2.8087$.

Using this, we gain that the exponent of the first term of (42) is strictly less than -1 for $c > 0.9684$, and so if $c > 0.9684$, E will not occur for any system (a, b, β, Y) with high probability, and so, with high probability, G will be connected. \square

5 Conclusion and Open Questions

In the last section we worked quite hard to bring the bound for the connectivity threshold down below $\log n$. However, the bound we proved, $0.9684 \log n$, is actually lower than the previously best known bound for the directed model of $0.9967 \log n$ proved in [6], and so since the edge in our strict undirected model are exactly the bidirectional edges in the connected model, it improves the bound for the directed model as well.

In fact, we believe a much stronger result holds. It seems that in both the directed model and strict undirected model the barrier to connectivity is an isolated vertex (or at least a very concentrated cluster of sub-logarithmic size). If this is the case, then it seems likely that the connectivity threshold for both models is the same (this does not immediately follow from the barrier in both

cases being an isolated vertex, since in the directed model the isolated vertex is in an in-component by itself, where as it may be possible that an isolated point in the strict undirected model has in-edges, but not from any of its k -nearest neighbours, however set-ups where this occurs seem less likely than an isolated vertex in an in-component).

In fact, the lower bound proved on the connectivity threshold for both models is essentially the threshold for having a point with no in-edges, and so putting this all together motivates the following conjecture:

Conjecture 1. *The barrier for connectivity for both the directed model and the strict undirected model, is an isolated vertex (or concentrated cluster of sub-logarithmic size) with no in-edges, and so the connectivity threshold in both models is the same (and something a little over $0.7209 \log n$).*

It is possible to strengthen the bounds of several of the results proved in this paper (although with a fair amount of extra work). The upper bound on the size of a small component around the connectivity threshold of $0.309 \log n$ (Lemma 34) can be improved to $0.203 \log n$ by using a stronger version of Lemma 33 (although the conditions needed to apply it then require more work to check).

The bound on the threshold for the edges of different components crossing (Theorem 1) can also be improved significantly. By determining the exact positions of a_1 and a_2 that maximise the ratio $|H|/|H \cup L|$ the bound can be reduced to around $0.5 \log n$, although this is almost certainly still a long way off the actual threshold.

A Definitions and Notation from Section 3.2

We collate here all the definitions and notation used in Section 3.2 in the order in which they appear.

- We say that a_1, a_2, b_1 and b_2 form a *crossing pair* if there are two different components X and Y with $a_1, a_2 \in X, b_1, b_2 \in Y$ and the straight line segments a_1a_2 and b_1b_2 intersect and are both in the graph G , such that $\|a_1a_2\| \leq \|b_1b_2\|$, $\|a_1b_1\| \leq \|a_1b_2\|$ and $d(a_1, b_1b_2) \leq d(a_2, b_1b_2)$.
- For $i = 1, 2$, $r_i = \min\{\|a_i b_1\|, \|a_i b_2\|\}$ (so that $r_1 = \|a_1 b_1\|$).
- For $i = 1, 2$, $A_i = D_{a_i}(r_i)$.
- For $i = 1, 2$, $B_i = D_{b_i}(1)$.
- $w = (\frac{1}{2}, \frac{1}{2\sqrt{3}})$.
- T is the triangle with vertices b_1, b_2 and w .
- S_1 is the region $T \setminus (D_{b_1}(\frac{1}{2}) \cup D_{b_2}(\frac{1}{2}))$.
- $z = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$.
- T_2 is the triangle with vertices b_1, b_2 and z .
- S_2 is the region $T_2 \cap A_1 \cap \{x \in S_n : x\hat{b}_1b_2 > \frac{\pi}{6} \text{ and } x\hat{b}_2b_1 > \frac{\pi}{6}\}$.

- R_1 is the region $D^k(a_1) \cap (B_1 \setminus B_2)$ and R_2 is the region $D^k(a_1) \cap (B_2 \setminus B_1)$.
- For $i = 1, 2$, E_i is the elliptical region $\{x \in S_n : \|b_i x\| + \|a_1 x\| \leq 1\}$. We write $E_i(a_1)$ for this ellipse when a_1 is specified.
- For $i = 1, 2$, F_i is the elliptical region $\{x \in S_n : \|b_i x\| + \|a_2 x\| \leq 1\}$. We write $F_i(a_1)$ for this ellipse when a_2 is specified.
- For a set $S \subset S_n$, we write S^+ for the part of S which lies above the line through b_1 and b_2 , and S^- for the part of S which lies below the line b_1 and b_2 .
- M for the region $D^k(a_1) \cap D^k(a_2)$.
- $L_1 = (D^k(a_1) \cap E_1 \cap D_{b_1}(1/2)) \setminus M$.
- $L_2 = (D^k(a_1) \cap E_2 \cap D_{b_2}(1/2)) \setminus M$.
- $L_3 = M^+ \cap D_{b_1}(1/2) \cap D_{b_2}(1/2)$.
- $L_4 = T_2 \cap D^k(a_2) \cap \{x : x\hat{b}_1 b_2 \leq \pi/6 \text{ or } x\hat{b}_2 b_1 \leq \pi/6\}$.
- $L_5 = (D^k(a_2) \cap F_1 \cap D_{b_1}(1/2)) \setminus T_2$.
- $L_6 = (D^k(a_2) \cap F_2 \cap D_{b_2}(1/2)) \setminus T_2$.
- $H_1 = R_1 \setminus L_1$.
- $H_2 = R_2 \setminus L_2$.
- $H_3 = A_2 \setminus (B_1 \cup B_2)$.
- $H_4 = M^+ \setminus L_3$.
- $H = S_2 \cup \bigcup_{i=1}^4 H_i$.
- $L = \bigcup_{i=1}^6 L_i$.
- $v^+ = (\frac{3}{4}, \frac{\sqrt{3}}{4})$.
- $v^- = (\frac{3}{4}, -\frac{\sqrt{3}}{4})$.
- $u^+ = (\frac{1}{4}, \frac{\sqrt{3}}{4})$.
- $u^- = (\frac{1}{4}, -\frac{\sqrt{3}}{4})$.
- $w' = (\frac{1}{2}, -\frac{1}{2\sqrt{3}})$.
- For $i = 1, 2$, ρ_i is the radius of $D^k(a_i)$.

References

- [1] P. Balister, B. Bollobás, A. Sarkar and M. Walters, *Connectivity of random k -nearest neighbour graphs*, Advances in Applied Probability, **37**(1):1–24 (2005)
- [2] P. Balister, B. Bollobás, A. Sarkar and M. Walters, *A critical constant for the k -nearest neighbour model*, Advances in Applied Probability, **41**(1):1–12 (2009)
- [3] R. J. Gardner, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. (NS), **39** (3) (2002), 355–405
- [4] E. N. Gilbert, *Random Plane Networks*, Journal of the Society for Industrial Applied Mathematics **9** (1961), 533–543.
- [5] M.D. Penrose, *The longest edge of the random minimal spanning tree*, Annals of Applied Probability **7** (1997), 340–361.
- [6] M. Walters, *Small components in k -nearest neighbour graphs*. Preprint.
- [7] F. Xue and P. R. Kumar, *The number of neighbors needed for connectivity of wireless networks*. Wireless Networks **10** (2004), 169–181